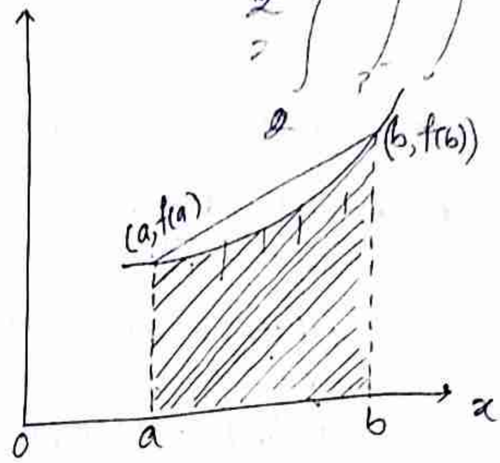
Numerical Integration :

$$I = \int_a^b f(x)\,dx$$

- **Trapezoidal Rule :**

Let us consider a curve $y = f(x)$, $a \leq x \leq b$, be approximated by the line joining the points $P(a, f(a))$ and $Q(b, f(b))$.



Using Newton's forward difference formula, the linear polynomial approximation to $f(x)$, interpolating at $(a, f(a))$ and $(b, f(b))$, is given by

$$f(x) = f(x_0) + \frac{(x-x_0)}{h} \nabla f(x_0) \longrightarrow (1)$$

Here $x_0 = a$

$h = (b-a)$

$x_1 = b$

$$\therefore \quad I = \int_a^b f(x)\,dx$$

$$= \int_a^b f(x_0)\,dx + \int_a^b \frac{(x-x_0)}{h} \nabla f(x_0)\,dx$$

$$= f(x_0)\int_a^b dx + \frac{1}{h}\left[\int_a^b (x-x_0)\,dx\right] \Delta f(x_0)$$

$$= f(x_0)(b-a) + \frac{1}{h}\left[\frac{(x-x_0)^2}{2}\right]_a^b \Delta f(x_0)$$

$$= f(x_0)(b-a) + \frac{1}{2h}\left[(b-x_0)^2 - (a-x_0)^2\right]\left[f(b) - f(a)\right]$$

$$= (b-a) f(a) + \frac{1}{2h}\left[(b-a)^2 - 0\right]\left[f(b) - f(a)\right]$$

$$= h\,f(a) + \frac{(b-a)^2}{2h}\left[f(b) - f(a)\right]$$

$$= h f(a) + \frac{h^2}{2h}\left[f(b) - f(a)\right]$$

$$= h\,f(a) + \frac{h}{2} f(b) - \frac{h}{2} f(a)$$

$$= \frac{h}{2} f(a) + \frac{h}{2} f(b)$$

$$I = \frac{h}{2}\left[f(a) + f(b)\right]$$

$\therefore$ The trapezoidal rule is given by

$$I = \int_a^b f(x)\,dx$$

$$= \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

$$\boxed{I = \frac{h}{2}\left[f(a) + f(b)\right]} \longrightarrow (2)$$

Geometrically, the RHS of the trapezoidal rule formula, is the area of the trapezoid with width $(b-a)$ and ordinates $f(a)$ and $f(b)$. This is an approximation of the area ~~of~~ under the curve $y = f(x)$ above x axis and between ordinates $x = a$ and $x = b$.

- Error Term in Trapezoidal rule :

$$\boxed{E(f,x) = -\frac{h^3}{12} f''(\xi)} \longrightarrow (3)$$

where $a \leqslant \xi \leqslant b$.

$\therefore$ The bound for the errors is given by

$$\left| E(f,x) \right| \leqslant \frac{(b-a)^3}{12} \left\{ \max_{a \leqslant x \leqslant b} \left| f''(x) \right| \right\} \longrightarrow (4)$$

If the length of the interval is large, then $(b-a)=h$ will also be large resulting large errors, which becomes meaningless. In such a case, the interval is devided into small subintervals, of equal length and then apply trapezoidal rule to evaluate the integral. This rule is called composite trape-zium rule.