Q. Derive the trapezoidal rule using Lagrange's linear interpolating polynomial.

Sol^n ~~The points~~

Lagrange's linear interpolation gives

$$f(x) = \frac{(x-b)}{(a-b)} f(a) + \frac{(x-a)}{(b-a)} f(b)$$

$$= \frac{-1}{(b-a)} \left\{ (\cancel{x}-b) f(a) + \frac{(x-a) f(b)}{(b-a)} \right\}$$

$$= \frac{1}{(b-a)} \left[(x-a) f(b) - (\cancel{x}-b) f(a) \right]$$

$$= \frac{1}{(b-a)} \left[x f(b) - a f(b) + b f(a) - x f(a) \right]$$

$$f(x) = \frac{1}{(b-a)}\left[x\{f(b)-f(a)\} + b f(a) - a f(b)\right]$$

$$\therefore \quad I = \int_a^b f(x)\,dx$$

$$= \frac{1}{(b-a)}\int_a^b x[f(b)-f(a)]\,dx + \frac{1}{(b-a)}\int_a^b [b f(a) - a f(b)]\,dx$$

$$= \frac{1}{(b-a)}(f(b)-f(a))\left[\frac{x^2}{2}\right]_a^b + \frac{1}{\cancel{(b-a)}}[b f(a) - a f(b)]\cancel{(b-a)}$$

$$= \frac{1}{2\cancel{(b-a)}} \times [f(b)-f(a)](b+a)\cancel{(b-a)}$$
$$+ [b f(a) - a f(b)]$$

$$= \frac{1}{2} b f(b) - \frac{1}{2} a f(a) + b f(a) - a f(b)$$

$$= \frac{1}{2} b f(b) - a f(b) + b f(a) - \frac{a}{2} f(a)$$

$$= f(b)\left(\frac{b}{2} - a\right) + f(a)$$

$$= \frac{1}{2}[f(b) - f(a)](b+a) + (b f(a) - a f(b))$$

$$= \frac{1}{2}\left[b f(b) - b f(a) + a f(b) - a f(a)\right.$$
$$\left. + b f(a) - a f(b)\right)$$

$$= \frac{b}{2} f(b) + \frac{a}{2} f(b) - \frac{b}{2} f(a) - \frac{a}{2} f(a)$$
$$+ b f(a) - a f(b)$$

$$= \frac{b}{2} f(b) - \frac{a}{2} f(b) + \frac{b}{2} f(a) - \frac{a}{2} f(a)$$

$$= \frac{f(b)}{2}(b-a) - \frac{b}{2}$$

$$= \frac{(b-a)}{2} f(b) + \frac{(b-a)}{2} f(a)$$

$\therefore \quad I = \frac{(b-a)}{2}\left[f(a) + f(b)\right] = \frac{h}{2}\left[f(a) + f(b)\right]$

which is the required Trapezoidal rule.

Q. Find the approximate value of $I = \int_0^1 \frac{dx}{(1+x)}$; using trapezoidal rule with 2, 4 and 8 equal subintervals.

Sol^n $\rightarrow$ $N=2$, $h = \frac{b-a}{N} = \frac{1-0}{2} = \frac{1}{2}$; 0, 0·5, 1

$N = 4$, $h = \frac{b-a}{N} = \frac{1-0}{4} = \frac{1}{4}$; 0, 0·25, 0·50, 1

$N = 8$, $h = \frac{b-a}{N} = \frac{1-0}{N} = \frac{1}{8}$;

0, 0·125, 0·25, 0·375, 0·5, 0·675, 0·75, 0·875, 1.0.

For composite trapezoidal rule for N intervals

$$I = \frac{h}{2}\left[f(x_0) + 2\{f(x_1) + f(x_2) + \cdots + f(x_{N-1})\} + f(x_N)\right]$$

and error $|E(f,x)| = +\frac{h^3}{12N^2} \max_{a \le x \le b} |f''(x)|$

$$N = \frac{(b-a)}{h}$$

For $N=2$, $I_1 = \frac{h}{2}\left[f(0) + 2f(0·5) + f(1·0)\right]$

$= \frac{h}{2}\left[1·0 + 2 \times 0·66667 + 0·5\right]$

$= 0·25\left[1·0 + 2 \times 0·66667 + 0·5\right]$

$= 0·708334.$

$N=4$; $I_2 = \frac{h}{2}\left[f(0) + 2\{f(0.25) + f(0.5) + f(0.75)\} + f(1.0)\right]$

$\Rightarrow I_2 = 0.125\left[1.0 + 2\{0.8 + 0.66667 + 0.571429\} + 0.5\right]$

$I_2 = 0.697024$

$N=8$, $I_3 = \frac{h}{2}\left[f(0) + 2\{f(0.125) + f(0.25) + f(0.375) + f(0.5) + f(0.625) + f(0.75) + f(0.875)\} + f(1.0)\right]$

$= 0.694122$

Q. Evaluate $\int_0^1 \frac{dx}{(x^2+6x+10)}$ using trapezoidal rule.

Sol^n

$f(x) = \frac{1}{x^2+6x+10}$ $\quad a=0$, $b=1$

$f(b) = \frac{1}{1+6+10}$

$f(b) = \frac{1}{17}$

and $f(a) = \frac{1}{0+0+10} = \frac{1}{10}$

$\therefore I = \frac{b-a}{2}\left[f(a) + f(b)\right]$

$= \frac{1-0}{2} \times \left(\frac{1}{10} + \frac{1}{17}\right)$

$$I = \frac{1}{2} \times \frac{17+10}{170}$$

$$= \frac{1}{2} \times \frac{27}{170}$$

$$I = 0.0794$$

Q. The velocity of a particle which starts from rest is given by the following table

t(sec)	0	2	4	6	8	10	12	14	16	18	20
v(ft/sec)	0	16	29	40	46	51	32	18	8	3	0

Calculate the total distance travelled in 20 seconds using Trapezoidal rule.

Solⁿ

We know that

Here $h = 2$

$a = 0$

$b = 20$

$f(x) = x$

$f(a) = 0$

$f(b) = 20$

$$v = \frac{ds}{dt}$$

$$\Rightarrow ds = v\,dt$$

$$\Rightarrow s = \int_0^{20} v\,dt$$

$\therefore N = \frac{b-a}{h}$

$= \frac{20-0}{2}$

$= 10$

$$\therefore I = \frac{(b-a)}{2}\left[\cancel{f(a)+f(b)}\right]$$

$$= \frac{20}{2}\, v$$

$$\Rightarrow I = \frac{h}{2}\left[f(x_0) + 2\{f(x_1)+f(x_2)+\cdots f(x_{N-1})\} + f(x_N)\right]$$

$$\Rightarrow I = \frac{2}{2}\left[f(0) + 2\{f(2)+f(4)+f(6)+f(8)+f(10)+f(12)+f(14)+f(16)+f(18)\} + f(20)\right]$$

$$\Rightarrow I = 0 + 2\{16+29+40+46+51+32+18+3\} + 0$$

$$\Rightarrow I = 486 \text{ feet.}$$