Notes of Mechanics

Unit-III: Dynamics of Rigid Bodies

B. Sc Physics Semester I

As per Pondicherry University Syllabus

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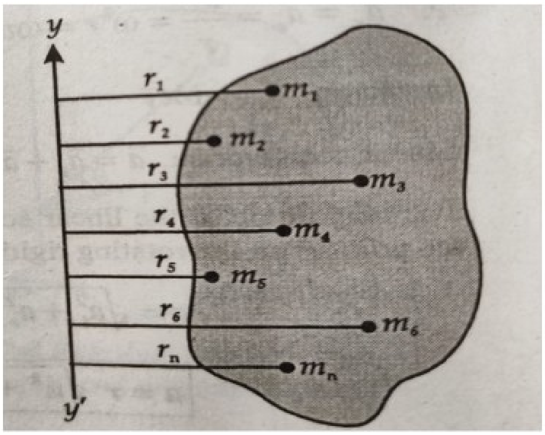
Degrees of freedom

Degree-of-freedom of a general mechanical system is defined as the minimum number of independent variables required to describe its configuration completely.

The set of variables (dependent or independent) used to describe a system are termed as the configuration variables. For a mechanism, these can be either Cartesian coordinates of certain points on the mechanism, or the joint angles of the links, or a combination of both.

Moment of Inertia

It is defined as sum of product of mass and square of distance from the axis of rotation.

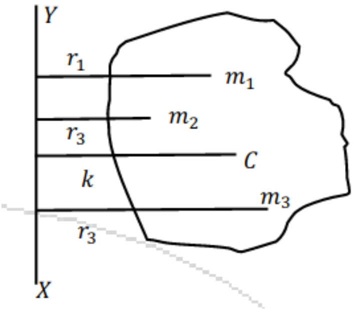


$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^3 + - - - - - - + m_n r_n^2$$

$$I = \sum mr^2 = \sum_{i=1}^{n} m_i r_i^2$$

For linear/translational motion, value of inertia depends only on the mass of the body. The kinetic energy in such motion depends on the mass and linear velocity of the body. But, when a body rotates about an axis, the kinetic energy of rotation is determined not only by its mass and angular velocity but also upon the position of the axis about which it rotates and distribution of mass about this axis.

Let us consider a body of mass rotating about an axis with angular velocity . All its particles have the same angular velocity but as they are at different distances from the axis of rotation, their linear velocities are different. Hence, we get



$$\begin{aligned} KE &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \cdots \\ &= \frac{1}{2}m_1 r_1^2 \omega^2 + \frac{1}{2}m_2 r_2^2 \omega^2 + \cdots \\ &= \frac{1}{2}\left(\sum mr^2\right)\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned}$$

where, I is the moment of inertia of the body.

Moment of inertia is scalar because its value about a given axis remains unchanged by reversing its direction of rotation about that axis. Greater the moment of inertia of a body, greater is the couple required to produce a given angular acceleration.

Physical significance of moment of inertia of a body about a certain axis :

The physical significance of the Moment of Inertia of a body about an axis is the allocation of the mass of the body in space about the axis. Greater the mass concentrated away from the axis, greater the moment of inertia. Moment of inertia plays the same role in rotational motion as that of mass in translatory motion. Thus the moment of inertia in the rotational motion is comparable to the mass in translational motion because it plays a similar function in rotational motion as the mass plays in translational motion. It can also be described as the determination of the resistance of a body to an exterior torque, the similar method mass is the measure of the resistance of a body to an exterior force. Also, to bring about a change in the state of rotation, torque has to be applied. Thus the moment of inertia in the rotational motion is analogous to the mass in translational motion because it plays the same role in rotational motion as the mass plays in translational motion.

Radius of gyration: If entire mass of body is concentrated at a point such that the kinetic energy of rotation is the same as that of the body itself, then the distance of the point from the axis of rotation is called radius of gyration.

$$I = Mk^2 = \sum mr^2$$

Where K is called radius of gyration.

$$k = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + - - - - + r_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n} r_i^2}{n}}$$

Thus, radius of gyration is equal to root mean square distance of the particle from the axis of rotation.

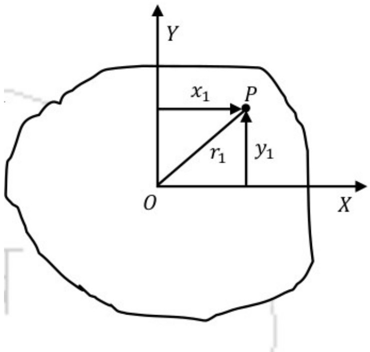
Theorems of Moment of Inertia

Theorem of perpendicular axes :

The moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moment of inertias of the lamina about the two axes at right angles to each other in its own plane intersecting each other at the point where the perpendicular axis passes through it.

$$I=I_X+I_Y$$

Let OX and OY be two perpendicular axes in the plane of the lamina. Let m_1 be the mass of a particle at point at distance r_1 from an axis through origin O, perpendicular to plane XOY.



Moment of inertia of the particle about X - axis = $m_1 y_1^2$
Moment of inertia of the particle about Y - axis = $m_1 x_1^2$

If we divide the whole lamina into a number of particles of masses

$m_1, m_2, m_3, ...$, at distances $r_1, r_2, r_3, ...$ from the axis.

Hence, the moment of inertia about X - axis and Y - axis are given as,

$$I_x=m_1 y_1^2+m_2 y_2^2+m_3 y_3^2+............=\sum my^2$$
$$I_Y=m_1 x_1^2+m_2 x_2^2+m_3 x_3^2+............=\sum mx^2$$

Moment of inertia of the lamina about a perpendicular axis through origin is given as,

$$I=m_1 r_1^2+m_2 r_2^2+m_3 r_3^2+............=\sum mr^2$$

$$I=m_1(x_1^2+y_1^2)+m_2(x_2^2+y_2^2)+m_3(x_3^2+y_3^2)+............$$

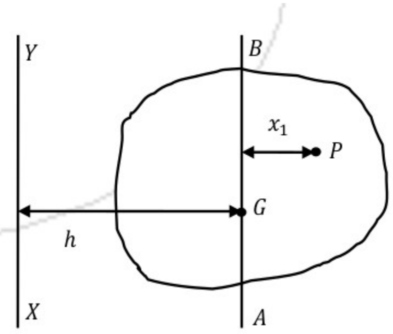
$$I=(m_1 x_1^2+m_2 x_2^2+m_3 x_3^2+m_4 x_4^2+....)+(m_1 y_1^2+m_2 y_2^2+m_3 y_3^2+m_4 y_4^2+....)$$

$$I=\sum m x^2+\sum m y^2$$

$$I=I_X+I_Y$$

Theorem of parallel axes :

The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity and the product of its mass and the square of the distance between the two axes.
Let XY be an axis in the plane of paper and AB is a parallel axis through G, the centre of mass of the body. The perpendicular distance between the two axes is h. Let M be the mass of the body and, m the mass of the element at P, at a distance x_1 from AB.



Moment of inertia of the element having mass m_1 about an axis XY is = $m_1(x_1+h)^2$ = $m_1(x_1^2+h^2+2x_1 h)$

Moment of inertia of the body about axis XY,

$$I=\sum m_i x_i^2+\sum m_i h^2+\sum 2m_i x_i h$$
$$I=I_G+\left(\sum m_i\right)h^2+2h\left(\sum m_i x_i\right)$$

where $I_G = \Sigma m_i x_i^2$ be the moment of inertia of the body about AB, passing through G.

Similarly, $\Sigma m_i x_i$ is sum of the moments of all the particles about AB, passing through G.

Since the body is balanced about the centre of mass G, algebraic sum of all the moments about G is zero.

$$I = I_G + \left(\sum m_i\right) h^2$$

$$I = I_G + Mh^2$$

Moment of Inertia of a circular disc

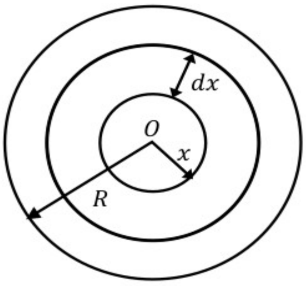
Circular Disc

a) About an axis through the centre of the disc perpendicular to its plane

Let us consider a circular disc of radius R. Let us consider an elementary ring of radius x and width dx. The area of the disc is given as $2\pi x dx$. Mass per unit area of the circular disc is $M/_{\pi R^2}$. Hence, mass of the elementary ring is given as,

$$= \frac{M}{\pi R^2} 2\pi x \, dx$$

$$= \frac{2M}{R^2} x \, dx$$



Moment of inertia of the element about an axis through its centre perpendicular to its plane is given as,

$$= \frac{2M}{R^2} x \, dx \, . \, x^2$$

$$= \frac{2M}{R^2} x^3 \, dx$$

Hence, the moment of inertia is given as,

$$\boldsymbol{I} = \frac{2M}{R^2} \int_0^R x^3 \, dx = \boldsymbol{\frac{1}{2} M R^2}$$

b) About the diameter of the circular disc

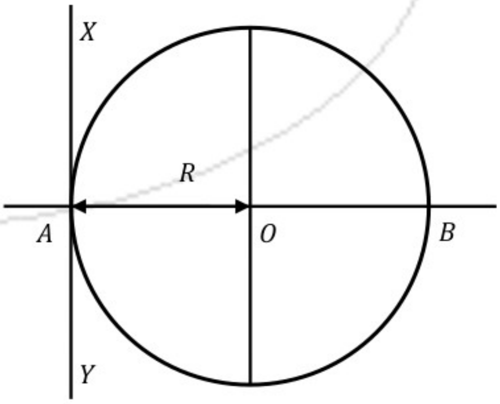
Let us consider AB to be the diameter. Let I_A and I_B be the moment of inertia about the end points of the diameter. Therefore,

$$I = I_A + I_B$$

By the symmetry of the figure, $I_A = I_B$. Hence,

$$I = 2I_A$$

$$\Rightarrow \quad \boldsymbol{I_A} = \frac{I}{2} = \boldsymbol{\frac{1}{4} M R^2}$$



c) About the tangent of the circular disc

Let XY be a tangent at A.

By the theorem of parallel axes,

$$I_T = I_A + Mh^2$$

$$I_T = I_A + M(OA)^2$$

$$I_T = \frac{1}{4} M R^2 + M R^2$$

$$\boldsymbol{I_T} = \boldsymbol{\frac{5}{4} M R^2}$$

Solid cylinder

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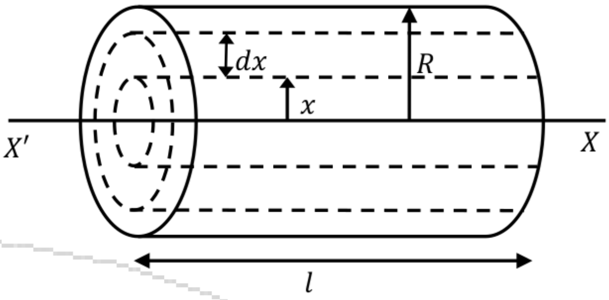
a) About its own axis of symmetry



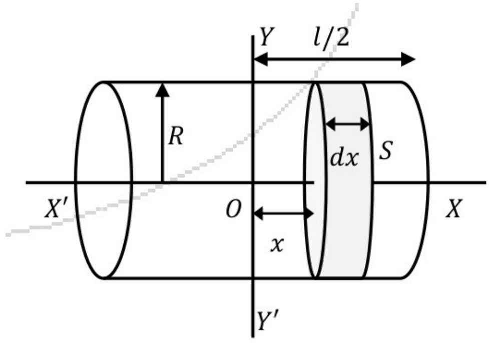
Let us consider a solid cylinder of mass M, length l and radius R. The volume of the cylinder is $\pi R^2 l$. Hence, the mass density of the cylinder is $\frac{M}{\pi R^2 l}$ cylinder of width coaxial cylinder is. Now let us consider a coaxial cylinder of width dx, at distance x from the axis of symmetry XX'. Hence, the volume of the coaxial cylinder (2πxl dx). Hence, the mass of the coaxial cylinder is given as,

$$Mass\ of\ the\ coaxial\ cylinder=\frac{M}{\pi R^2 l}(2\pi x\, l\, dx)=\frac{2M}{R^2}x\,dx$$

Therefore, Moment of inertia of the coaxial cylinder = $\frac{2M}{R^2}(x\,dx)x^2=\frac{2M}{R^2}x^3\,dx$

Hence, the moment of inertia = $\frac{2M}{R^2}\int_0^R x^3\,dx=\frac{1}{2}M\,R^2$

b) About the axis passing through the centre and perpendicular to its own axis of symmetry



Let XX' be the axis of symmetry and YY' be the axis perpendicular to XX'. Let us consider a circular disc S of width dx at a distance from YY' axis. Mass per unit length of the cylinder is (*M/l*). Hence the mass of the disc is $\frac{M}{l}dx$. Moment of inertia of this disc about the diameter of the rod = $\frac{M}{l}dx(\frac{R^2}{4})$

Moment of inertia of the disc about YY' axis given by parallel axis theorem = $(\frac{M}{l}dx)(\frac{R^2}{4})+(\frac{M}{l}dx)x^2$

Hence, the moment of inertia of the cylinder is given as,

$$I=\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{M}{l}(\frac{R^2}{4})dx+\int_{-\frac{l}{2}}^{\frac{l}{2}}\frac{M}{l}x^2\,dx$$

$$I=\frac{M}{l}(\frac{R^2}{4})2(\frac{l}{2})+\frac{M}{l}\frac{1}{3}2\frac{l^3}{8}$$

$$I=M[(\frac{R^2}{4})+(\frac{l^2}{12})]$$

For a thin rod, $R\approx 0$. Hence moment of inertia is given as,

$$I=\frac{Ml^2}{12}$$

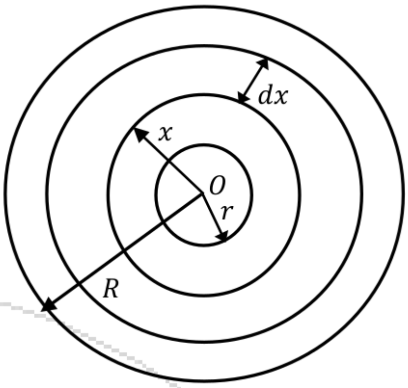
Annular Ring

a) About an axis passing through the origin and perpendicular to its plane

Let us consider a ring having inner radius r and outer radius R having mass M. Area of the face of the ring is $\pi(R^2 - r^2)$. Mass per unit area of the ring is given as, $^{M}/_{\pi(R^2-r^2)}$. Let us now consider a ring having radius x and $x + dx$. Face area of this ring is $2\pi x\ dx$. Mass of this ring is,

$$= \frac{M}{\pi(R^2 - r^2)} 2\pi x\ dx$$

$$= \frac{2M}{(R^2-r^2)} x\ dx$$



Moment of inertia of this ring is given as,

$$= \frac{2M}{R^2 - r^2} x\ dx\ .x^2$$

$$= \frac{2M}{R^2 - r^2} x^3\ dx$$

Hence, the moment of inertia is given as,

$$I = \frac{2M}{R^2 - r^2}\int_r^R x^3\ dx = \frac{2M}{R^2 - r^2}\left[\frac{R^4 - r^4}{4}\right]$$

$$\boldsymbol{I = \frac{M}{2}[R^2 + r^2]}$$

b) About its diameter

Let us consider AB to be the diameter. Let I_A and I_B be the moment of inertia about the end points of the diameter.

Therefore, $\boldsymbol{I = I_A + I_B}$

By the symmetry of the figure, $I_A = I_B$

$$I = 2I_A$$

$$I_A = \frac{I}{2} = \frac{1}{4}[R^2 + r^2]$$

Solid Sphere

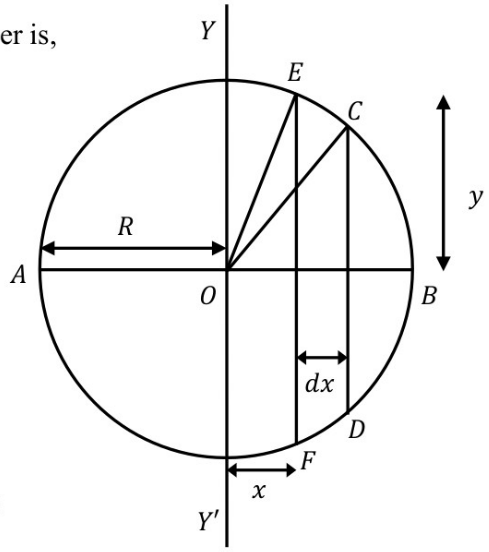
a) About its diameter

Let us consider a solid sphere of radius R and mass M. Consider a thin circular slice of radius, $y = \sqrt{R^2 - x^2}$. The volume of the slice is $\pi(R^2 - x^2)dx$. Let ρ be the mass per unit volume of the sphere. Hence, mass of the slice is given as, $\rho\pi(R^2 - x^2)dx$. Moment of inertia of this slice about a diameter AB is given as,

$$= \frac{1}{2}[\rho\pi(R^2 - x^2)dx]y^2$$

$$= \frac{1}{2}[\rho\pi(R^2 - x^2)^2 dx]$$

Hence, the moment of inertia of the disc about its diameter is,



$$I = \int_{-R}^{R} \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

$$I = 2 \times \frac{1}{2}\rho\pi \int_{0}^{R} (R^2 - x^2)^2 dx$$

$$I = \rho\pi \int_{0}^{R} (R^4 - 2R^2x^2 + x^4) dx$$

$$I = \rho\pi \left[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^R$$

$$I = \frac{8\rho\pi}{15}R^5$$

But mass of the sphere is $M = \frac{4}{3}\pi R^3\rho$. Hence, the above expression can be written as,

$$I = 2\left(\frac{4}{3}\pi R^3\rho\right)\left(\frac{R^2}{5}\right)$$

$$\boldsymbol{I = \frac{2}{5}MR^2}$$

b) About a tangent

Let XY be a tangent at A.

By the theorem of parallel axes,

$$I_T = I_A + Mh^2$$

$$I_T = I_A + M(OA)^2$$

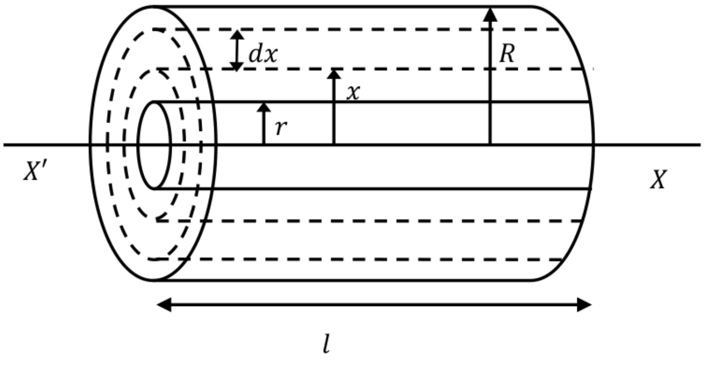
$$I_T = \frac{2}{5}MR^2 + MR^2$$

$$\boldsymbol{I_T = \frac{7}{5}MR^2}$$

Hollow Cylinder

a) About its own axis of symmetry

Let us consider a hollow cylinder of mass M, length l and inner radius r and outer radius R. The volume of the cylinder is given as, $\pi(R^2 - r^2)l$. Hence, the mass density of the cylinder is $M/\pi(R^2 - r^2)l$. Now let us consider a coaxial cylinder of width dx at distance x from the axis of symmetry. Hence, the volume of the coaxial cylinder is $2\pi xl\ dx$. Hence, the mass of the coaxial cylinder is given as,



$$= \frac{M}{\pi(R^2 - r^2)l} 2\pi xl\ dx$$

$$= \frac{2M}{R^2 - r^2} x\ dx$$

Moment of inertia of the coaxial cylinder is,

$$= \left(\frac{2M}{R^2 - r^2} x\, dx\right) x^2$$

$$= \left(\frac{2M}{R^2 - r^2} x^3\, dx\right)$$

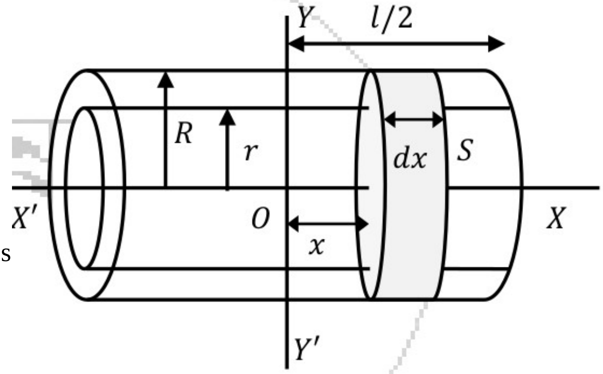
Hence, the moment of inertia is given as,

$$I = \frac{2M}{R^2 - r^2}\int_r^R x^3\, dx = \frac{2M}{R^2 - r^2}\left[\frac{R^4 - r^4}{4}\right]$$

$$\boldsymbol{I = \frac{M}{2}[R^2 + r^2]}$$

b) About the axis passing through the centre and perpendicular to its own axis of symmetry

Let XX' be the axis of symmetry and YY' be the axis perpendicular to XX'. Let us consider a circular disc S of width dx at a distance x from YY' axis. Mass per unit length of the cylinder is M/l. Hence the mass of the disc is $\frac{M}{l}dx$. Moment of inertia of this disc about the diameter of the rod is,



$$= \left(\frac{M}{l}dx\right)\frac{R^2 + r^2}{4}$$

Moment of inertia of the disc about YY’ given by parallel axes theorem, is given by

$$= \left(\frac{M}{l}dx\right)\frac{R^2 + r^2}{4} + \left(\frac{M}{l}dx\right)x^2$$

Hence, the moment of inertia of the cylinder is given as,

$$I = \int_{-l/2}^{l/2}\left(\frac{M}{l}dx\right)\frac{R^2 + r^2}{4} + \int_{-l/2}^{l/2}\left(\frac{M}{l}dx\right)x^2$$

$$I = \frac{M}{l}\left[\int_{-\frac{l}{2}}^{\frac{l}{2}}\left(\frac{R^2 + r^2}{4} + x^2\right)dx\right]$$

$$I = \frac{M}{l}\left[\frac{(R^2 + r^2)x}{4} + \frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$\boldsymbol{I = M\left[\frac{R^2 + r^2}{4} + \frac{l^2}{12}\right]}$$

(*a*) *Kinetic energy of a body about an axis through its centre of mass.*—Suppose we have a body of mass M rotating about an axis AB, passing through its centre of mass O, (Fig. 50). It, obviously, possesses kinetic energy due to its motion ; this energy of the body is called its *energy of rotation*, because it is due to its motion of rotation.

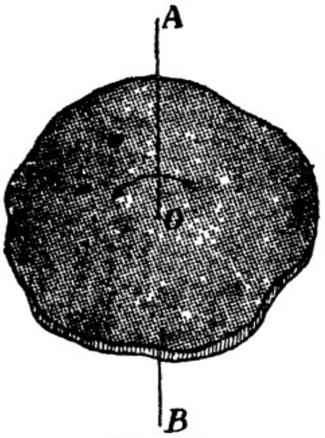


Fig. 50.

Imagine the body to be divided up into a large number of small particles, of masses m_1, m_2, m_3, etc., at distances r_1, r_2, r_3 .... etc., respectively from the axis AB. Then, we have

$$\text{linear velocity of } m_1 = r_1\omega = v_1 ;$$

of $m_2 = r_2\omega = v_2$; of $m_3 = r_3\omega = v_3$ and so on,

$\therefore$ *kinetic energy* of mass $m_1 = \frac{1}{2}m_1v_1^2$; of mass $m_2 = \frac{1}{2}m_2v_2^2$; of mass $m_3 = \frac{1}{2}m_3v_3^2$ and so on.

Or, total $K.E.$ of the body $= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \ldots\ldots$

$$= \tfrac{1}{2}\omega^2[m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots]$$

$$= \tfrac{1}{2}\omega^2\Sigma mr^2 = \tfrac{1}{2}\omega^2MK^2. \quad [\because \Sigma mr^2 = MK^2.$$

Or, **K.E. of the body** $= \frac{1}{2}\mathbf{MK^2}\omega^2 = \frac{1}{2}\mathbf{I}\omega^2$, $\quad [\because MK^2 = I.$

where I is the *moment of inertia* of the body about axis AB.

Now, if $\omega = 1$, then, obviously, $K.E.$ of the body $= \frac{1}{2} I$.

Or, $$I = 2\ K.E.$$

Thus, the moment of inertia of a body, rotating with unit angular velocity, is equal to twice its kinetic energy of rotation.

(*b*) *K.E. of body which is not only rotating but whose centre of mass has also a linear velocity* v.—A body which is rotating as well as moving forwards with a velocity v, has both types of kinetic energy, *viz.*, (*i*) *energy of rotation*, because of its motion of rotation about a perpendicular axis through its centre of mass, and (*ii*) *energy of translation*, because of its linear motion. And, clearly, therefore, we have $K.E.$ *of rotation* of the body $= \frac{1}{2} I\omega^2$,

and its $K.E.$ *of translation* $= \frac{1}{2} Mv^2$.

$\therefore$ *total K.E. of the body* $= K.E.$ *of rotation* $+ K.E.$ *of translation*,

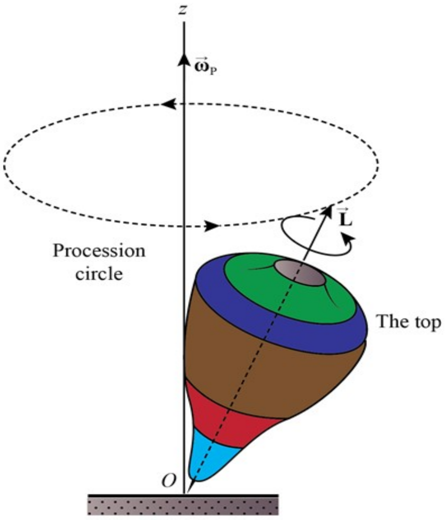
$$= \tfrac{1}{2} I\omega^2 + \tfrac{1}{2}Mv^2 = \tfrac{1}{2}MK^2\omega^2 + \tfrac{1}{2}Mv^2.$$

$$= \tfrac{1}{2}M(K^2\omega^2 + v^2) = \tfrac{1}{2}MK^2(v^2/r^2) + \tfrac{1}{2}Mv^2$$

because $\omega^2 = v^2/r^2$ where r is the radius of the body.

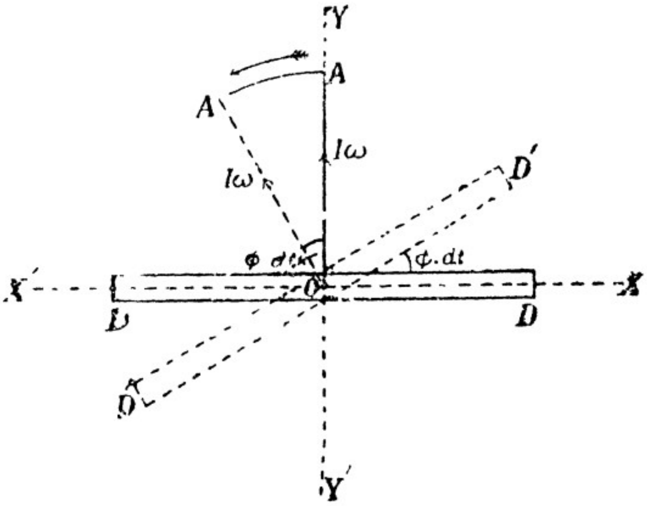
Or, *total kinetic energy of the body* $= \frac{1}{2} Mv^2[(K^2/r^2) + 1]$.

Precessional motion (qualitative only)



In rotational motion, we may have a constant angular acceleration acting on a body having a constant angular speed. This is rendered possible by the plane of rotation changing direction at a given rate , without, in any way, affecting the rate of rotation of the body about its axis of rotation, or axis of spin, as it is also sometimes referred to. This change in the plane of rotation is called 'precession', and is caused by a couple or torque, called the precessional torque, acting: in a plane, perpendicular to the immediate or instantaneous plane of rotation (or spin) of the body.

Let us consider that DD, be the edge of a disc, with its plane revolving about its geometric axis, with an angular velocity ω. Then, if its moment of inertia about this axis be I, its angular momentum will Iω. Let this be represented by the straight line OA, clearly be lw. drawn perpendicular to the plane of rotation of the disc. Now, let the axle of the disc also rotate, i.e., let there be a precessional motion, about an axis, perpendicular to the plane of the paper at a (precessional) rate Φ so that, after a small interval of time dt, the disc takes up the position DD', making an angle Φ.dt with its original position. Its angular momentum, again equal to Iω is now represented by the straight line OA'.



The change in the angular momentum of the disc is thus represented vectorially by

$$AA' = I\omega\,(\Phi.dt)$$

This change has, clearly, been brought about in time dt, and therefore

Rate of change of momentum of the disc = $I\omega\,(\Phi.dt)/dt = I\omega.\Phi$

And, since the rate of change of angular momentum of a rotating body is equal to the torque applied to it, we have

$T_1 = I\omega\Phi$

where T_1 is the torque applied to the disc.

So that, the rate of precession $\Phi = T_1/(I\omega)$

Gyroscope

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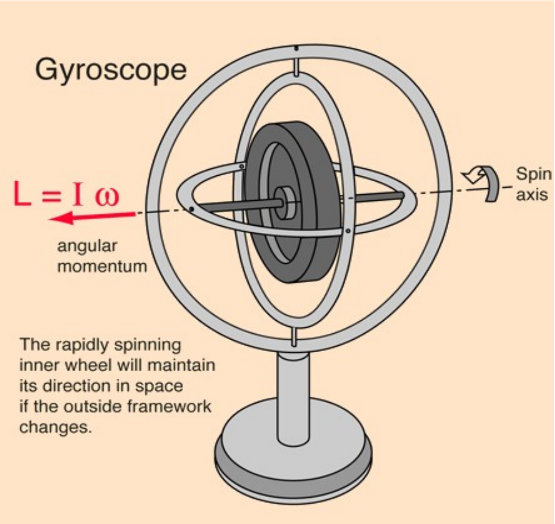
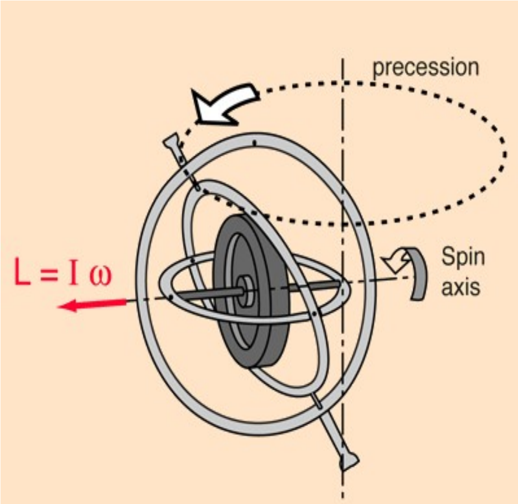
One typical type of gyroscope is made by suspending a relatively massive rotor inside three rings called gimbals. Mounting each of these rotors on high quality bearing surfaces insures that very little torque can be exerted on the inside rotor.

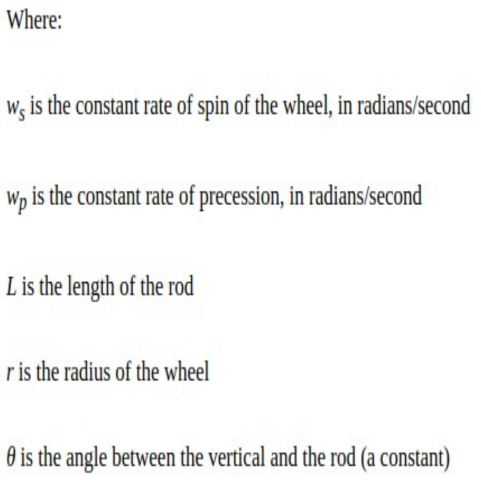




The classic image of a gyroscope is a fairly massive rotor suspended in light supporting rings called gimbals which have nearly friction less bearings and which isolate the central rotor from outside torques. At high speeds, the gyroscope exhibits extraordinary stability of balance and maintains the direction of the high speed rotation axis of its central rotor. The implication of the conservation of angular momentum is that the angular momentum of the rotor maintains not only its magnitude, but also its direction in space in the absence of external torque. The classic type gyroscope finds application in gyro-compasses, but there are many more common examples of gyroscopic motion and stability. Spinning tops, the wheels of bicycles and motorcycles, the spin of the Earth in space, even the behaviour of a boomerang are examples of gyroscopic motion.

If a gyroscope is tipped, the gimbals will try to reorient to keep the spin axis of the rotor in the same direction. If released in this orientation, the gyroscope will precess in the direction shown because of the torque exerted by gravity on the gyroscope.

Working of a gyroscope



Where:

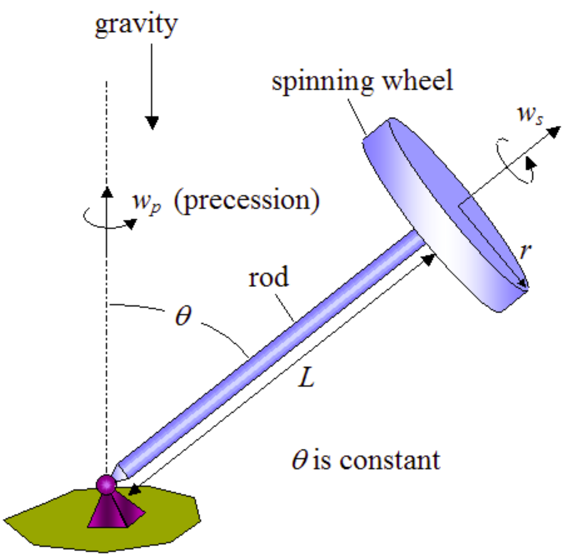
w_s is the constant rate of spin of the wheel, in radians/second

w_p is the constant rate of precession, in radians/second

L is the length of the rod

r is the radius of the wheel

θ is the angle between the vertical and the rod (a constant)

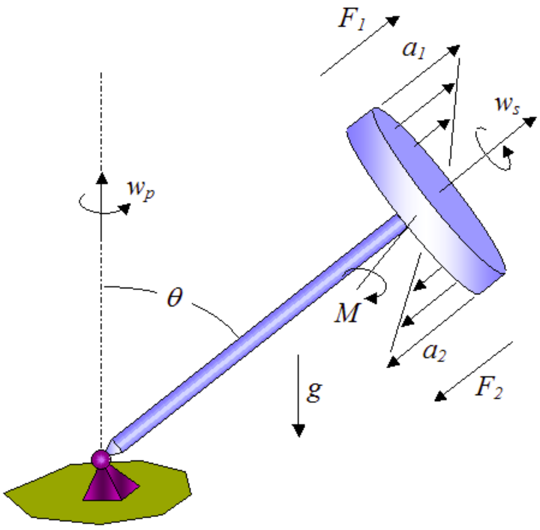


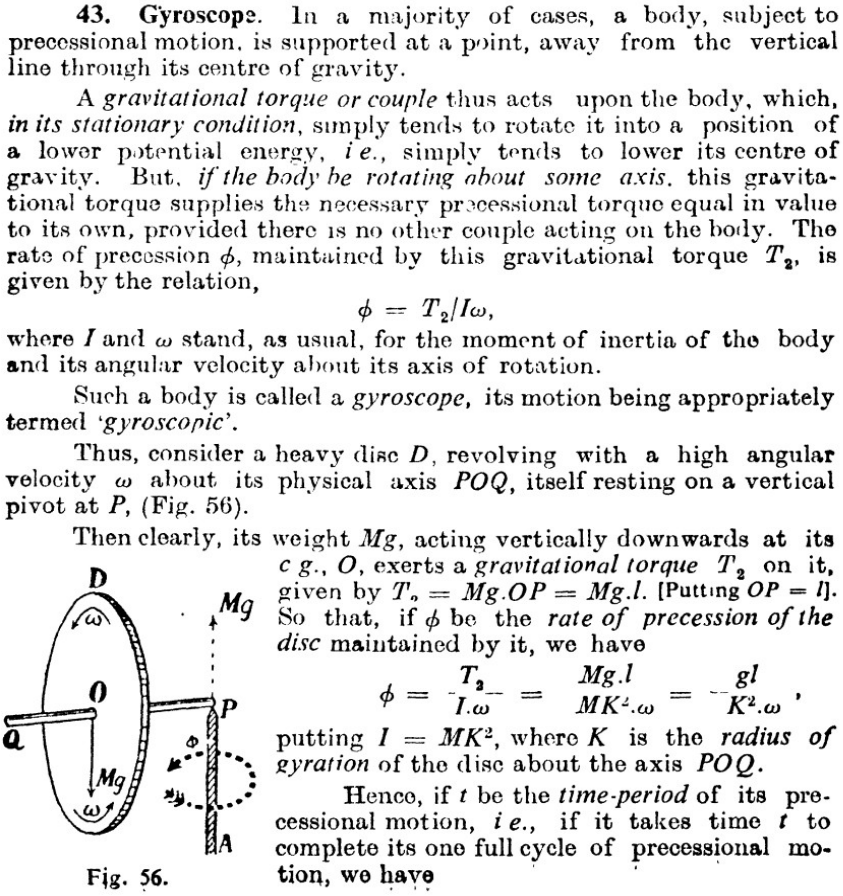
As the wheel spins at a rate w_s, the gyroscope precesses at a rate w_p about the pivot at the base (with θ constant).

The question is, why doesn't the gyroscope fall down due to gravity?

Due to the combined rotation w_s and w_p, the particles in the top half of the spinning wheel experience a component of acceleration a_1 normal to the wheel (with distribution as shown in the figure below), and the particles in the bottom half of the wheel experience a component of acceleration a_2 normal to the wheel in the opposite direction (with distribution as shown). Due to Newton’s second law, this means that a net force F_1 must act on the particles in the top half of the wheel, and a net force F_2 must act on the particles in the bottom half of the wheel. These forces act in opposite directions. Therefore a clockwise torque M is needed to sustain these forces. The force of gravity pulling down on the gyroscope creates the necessary clockwise torque M.

In other words, due to the nature of the kinematics, the particles in the wheel experience acceleration in such a way that the force of gravity is able to maintain the angle θ of the gyroscope as it precesses. This is the most basic explanation behind the gyroscope physics.





43. Gyroscope. In a majority of cases, a body, subject to precessional motion, is supported at a point, away from the vertical line through its centre of gravity.

A *gravitational torque or couple* thus acts upon the body, which, *in its stationary condition*, simply tends to rotate it into a position of a lower potential energy, *i e.*, simply tends to lower its centre of gravity. But, *if the body be rotating about some axis*, this gravitational torque supplies the necessary precessional torque equal in value to its own, provided there is no other couple acting on the body. The rate of precession ϕ, maintained by this gravitational torque T_2, is given by the relation,

$$\phi = T_2/I\omega,$$

where I and ω stand, as usual, for the moment of inertia of the body and its angular velocity about its axis of rotation.

Such a body is called a *gyroscope*, its motion being appropriately termed ‘*gyroscopic*’.

Thus, consider a heavy disc D, revolving with a high angular velocity ω about its physical axis POQ, itself resting on a vertical pivot at P, (Fig. 56).

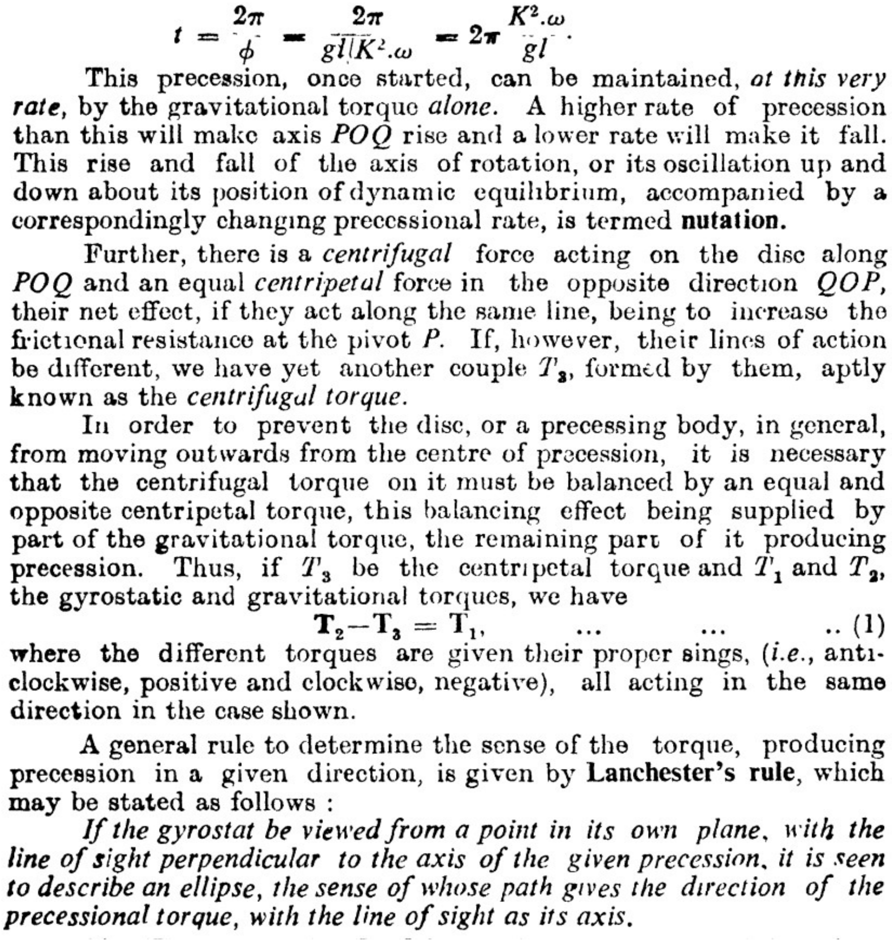
Then clearly, its weight Mg, acting vertically downwards at its c g., O, exerts a *gravitational torque* T_2 on it, given by $T_2 = Mg.OP = Mg.l$. [Putting $OP = l$]. So that, if ϕ be the *rate of precession of the disc* maintained by it, we have

$$\phi = \frac{T_2}{I.\omega} = \frac{Mg.l}{MK^2.\omega} = \frac{gl}{K^2.\omega},$$

putting $I = MK^2$, where K is the *radius of gyration* of the disc about the axis POQ.

Hence, if t be the *time-period* of its precessional motion, *i e.*, if it takes time t to complete its one full cycle of precessional motion, we have

Fig. 56.



$$t = \frac{2\pi}{\phi} = \frac{2\pi}{gl/K^2.\omega} = 2\pi\frac{K^2.\omega}{gl}.$$

This precession, once started, can be maintained, *at this very rate*, by the gravitational torque *alone*. A higher rate of precession than this will make axis *POQ* rise and a lower rate will make it fall. This rise and fall of the axis of rotation, or its oscillation up and down about its position of dynamic equilibrium, accompanied by a correspondingly changing precessional rate, is termed **nutation.**

Further, there is a *centrifugal* force acting on the disc along *POQ* and an equal *centripetal* force in the opposite direction *QOP*, their net effect, if they act along the same line, being to increase the frictional resistance at the pivot *P*. If, however, their lines of action be different, we have yet another couple T_3, formed by them, aptly known as the *centrifugal torque*.

In order to prevent the disc, or a precessing body, in general, from moving outwards from the centre of precession, it is necessary that the centrifugal torque on it must be balanced by an equal and opposite centripetal torque, this balancing effect being supplied by part of the gravitational torque, the remaining part of it producing precession. Thus, if T_3 be the centripetal torque and T_1 and T_2, the gyrostatic and gravitational torques, we have

$$\mathbf{T}_2 - \mathbf{T}_3 = \mathbf{T}_1, \qquad \ldots \qquad \ldots \qquad .. (1)$$

where the different torques are given their proper sings, (*i.e.*, anti-clockwise, positive and clockwise, negative), all acting in the same direction in the case shown.

A general rule to determine the sense of the torque, producing precession in a given direction, is given by **Lanchester's rule**, which may be stated as follows :

If the gyrostat be viewed from a point in its own plane, with the line of sight perpendicular to the axis of the given precession, it is seen to describe an ellipse, the sense of whose path gives the direction of the precessional torque, with the line of sight as its axis.

Centripetal force

According to Newton's first law of motion, a body must continue to move with a uniform velocity in a straight line, unless acted upon by a force. It follows, therefore, that when a body moves along a circle, some force is acting upon it, which continually deflects it from its straight or linear path and, since the body has an acceleration towards the centre, it is obvious that the force must also be acting in the direction of this acceleration, i.e., along the radius, or towards the centre of its circular path. It is called the centripetal force, and its value is given by the product of the mass of the body and its centripetal acceleration. Thus, if be the mass of the body, we have

$$\textbf{centripetal force} = \frac{mv^2}{r} = \frac{m\omega^2 r^2}{r} = m\omega^2 r$$

Centrifugal force

The equal and opposite reaction to the centripetal force is called the centrifugal force, because it tends to take the body away from the centre. Centripetal force and centrifugal force being just action and reaction in the sense of Newton's third law of motion.

Thus, in the case of a stone, whirled round at the end of a string, not only is the stone acted upon by a force, (the centripetal force) along the string towards the centre, but the stone also exerts an equal and opposite force along the string (the centrifugal force), on the other hand, away from the centre, also along the string.

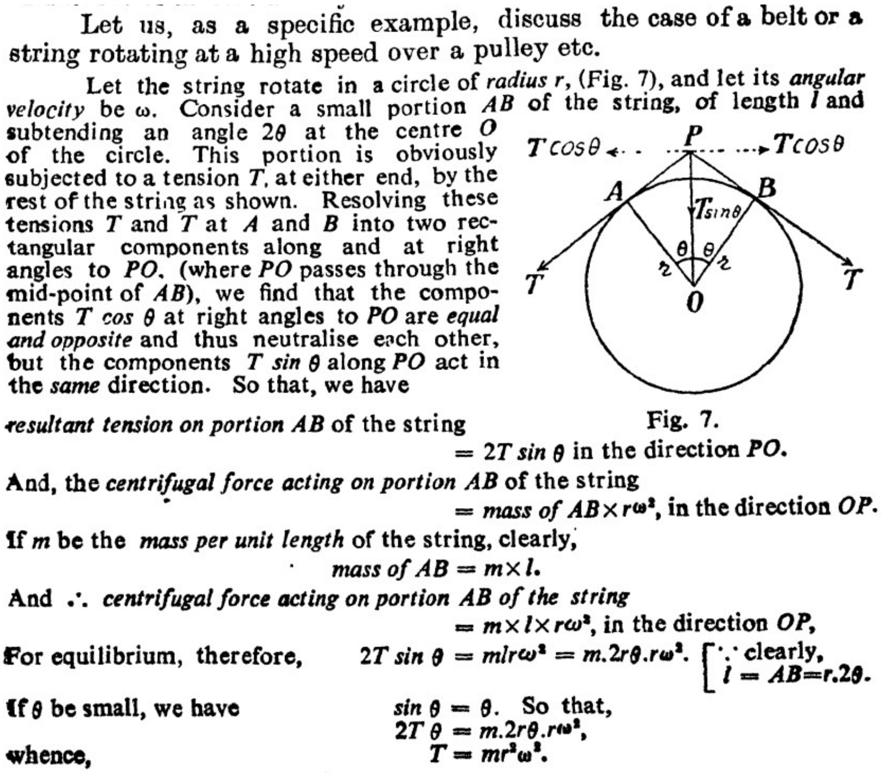
Coriolis forces

(Already discussed in class.)

Practical Applications of Centripetal and Centrifugal forces

1. Road Curves : The centripetal force being directly proportional to the square of the linear velocity of the body and inversely proportional to the radius of its circular path, the radii of curvature of road curves must be large and the speed of the vehicles negotiating, them slowed in order to keep , the value of the centripetal force required within reasonable limits.

2. Rotating Machinery : The centrifugal force being proportional to n^2 where n is the number of rotations made by the body per second. the spokes of a wheel, joining its outer revolving parts to the axis of rotation, experience an outward force, away from the centre, and are, therefore, in a state of tension, and may give way if the value of n is very large. So is the case with the other parts of rotating machinery, connecting its outer revolving part to the axis of rotation. In other words, there is a limit set to the value of n by the tension these connecting parts can withstand. This fact is, always kept in view while designing highly rotating machinery, like armatures of motors and dynamos etc.



Let us, as a specific example, discuss the case of a belt or a string rotating at a high speed over a pulley etc.

Let the string rotate in a circle of *radius r*, (Fig. 7), and let its *angular velocity* be ω. Consider a small portion *AB* of the string, of length *l* and subtending an angle 2θ at the centre *O* of the circle. This portion is obviously subjected to a tension *T*, at either end, by the rest of the string as shown. Resolving these tensions *T* and *T* at *A* and *B* into two rectangular components along and at right angles to *PO*, (where *PO* passes through the mid-point of *AB*), we find that the components *T cos θ* at right angles to *PO* are *equal and opposite* and thus neutralise each other, but the components *T sin θ* along *PO* act in the *same* direction. So that, we have

Fig. 7.

resultant tension on portion AB of the string

$$= 2T\sin\theta \text{ in the direction } PO.$$

And, the *centrifugal force acting on portion AB* of the string

$$= \text{mass of } AB \times r\omega^2, \text{ in the direction } OP.$$

If *m* be the *mass per unit length* of the string, clearly,

$$\text{mass of } AB = m \times l.$$

And ∴ *centrifugal force acting on portion AB of the string*

$$= m \times l \times r\omega^2, \text{ in the direction } OP,$$

For equilibrium, therefore, $2T\sin\theta = mlr\omega^2 = m.2r\theta.r\omega^2$. $\left[\because \text{ clearly, } l = AB = r.2\theta.\right.$

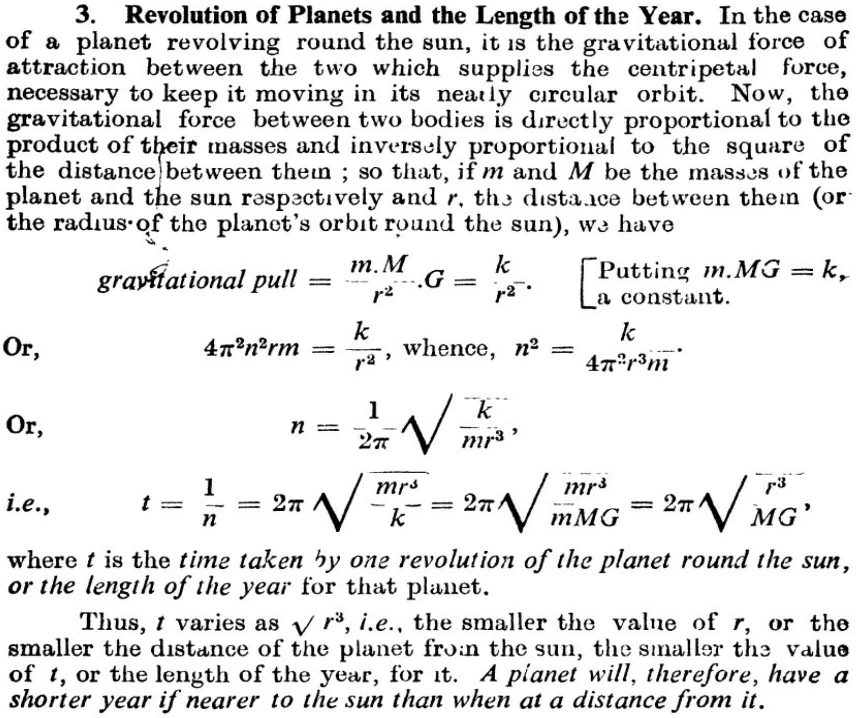
If θ be small, we have $\sin\theta = \theta$. So that,

$$2T\theta = m.2r\theta.r\omega^2,$$

whence,

$$T = mr^2\omega^2.$$

It will thus be seen that due to the centrifugal force, the tension in the string is very High. Indeed, if the rapidly rotating chain or belt be pushed off the pulley, it will run along like a rigid hoop.



3. Revolution of Planets and the Length of the Year. In the case of a planet revolving round the sun, it is the gravitational force of attraction between the two which supplies the centripetal force, necessary to keep it moving in its nearly circular orbit. Now, the gravitational force between two bodies is directly proportional to the product of their masses and inversely proportional to the square of the distance between them ; so that, if *m* and *M* be the masses of the planet and the sun respectively and *r*, the distance between them (or the radius of the planet's orbit round the sun), we have

$$\text{gravitational pull} = \frac{m.M}{r^2}.G = \frac{k}{r^2}. \quad \left[\begin{array}{l}\text{Putting } m.MG = k,\\ \text{a constant.}\end{array}\right.$$

Or, $$4\pi^2n^2rm = \frac{k}{r^2}, \text{ whence, } n^2 = \frac{k}{4\pi^2r^3m}.$$

Or, $$n = \frac{1}{2\pi}\sqrt{\frac{k}{mr^3}},$$

i.e., $$t = \frac{1}{n} = 2\pi\sqrt{\frac{mr^3}{k}} = 2\pi\sqrt{\frac{mr^3}{mMG}} = 2\pi\sqrt{\frac{r^3}{MG}},$$

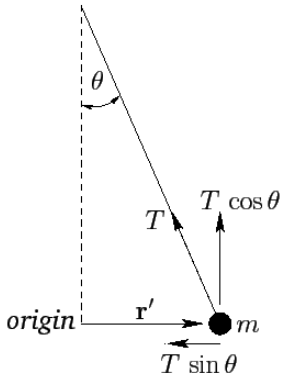
where *t* is the *time taken by one revolution of the planet round the sun, or the length of the year* for that planet.

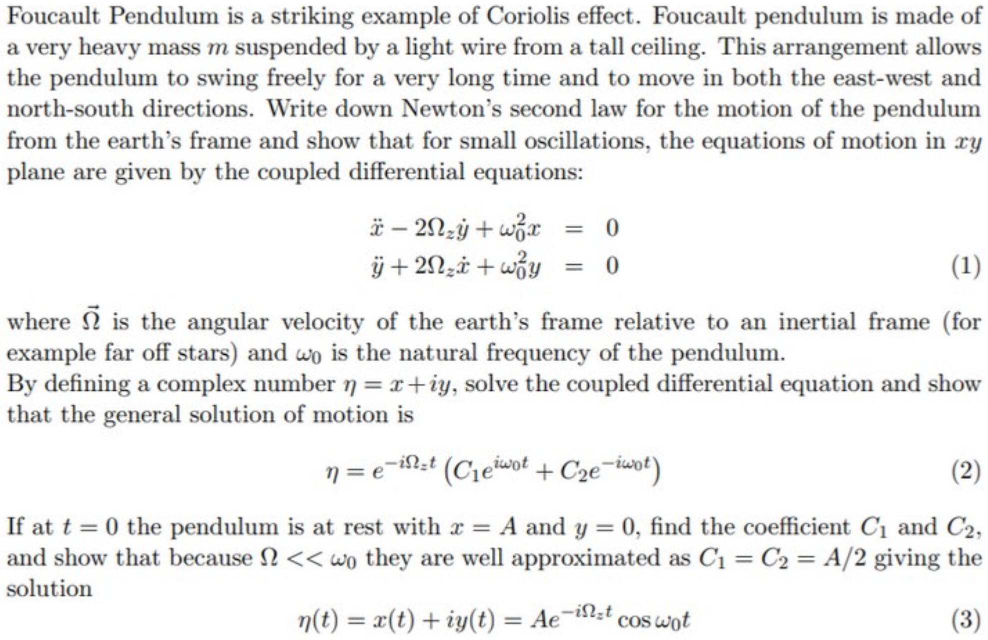
Thus, *t* varies as $\sqrt{r^3}$, *i.e.*, the smaller the value of *r*, or the smaller the distance of the planet from the sun, the smaller the value of *t*, or the length of the year, for it. *A planet will, therefore, have a shorter year if nearer to the sun than when at a distance from it.*

Foucault pendulum

A Foucault pendulum, or Foucault's pendulum, named after the French physicist Léon Foucault, was conceived as an experiment to demonstrate the rotation of the Earth; its action is a result of the Coriolis effect. It is a tall pendulum free to oscillate in any vertical plane and ideally should include some sort of motor so that it can run continuously rather than have its motion damped by air resistance.

The pendulum swings in a fixed plane and the Earth rotates beneath it, but this explanation is misleading. At the north or south pole, the pendulum is moving in a fixed plane (if we disregard the fact that the Earth is also revolving through space), so the plane of the pendulum seems to rotate through 360° as the Earth makes one full rotation. At any other point on Earth, however, the point at which the pendulum is attached cannot be considered a "fixed point," because that point also moves as the Earth rotates. The plane in which the pendulum swings is similarly in motion. Because of this, the amount of time that it takes for the pendulum to make one full rotation (with respect to its surroundings) is equal to one sidereal day (23.93 hours) divided by the sine of the latitude of its location. Since sin(0)=0, the plane of a pendulum located at the equator will not appear to move at all.





Foucault Pendulum is a striking example of Coriolis effect. Foucault pendulum is made of a very heavy mass m suspended by a light wire from a tall ceiling. This arrangement allows the pendulum to swing freely for a very long time and to move in both the east-west and north-south directions. Write down Newton's second law for the motion of the pendulum from the earth's frame and show that for small oscillations, the equations of motion in xy plane are given by the coupled differential equations:

$$\begin{aligned} \ddot{x} - 2\Omega_z\dot{y} + \omega_0^2 x &= 0 \\ \ddot{y} + 2\Omega_z\dot{x} + \omega_0^2 y &= 0 \end{aligned} \qquad (1)$$

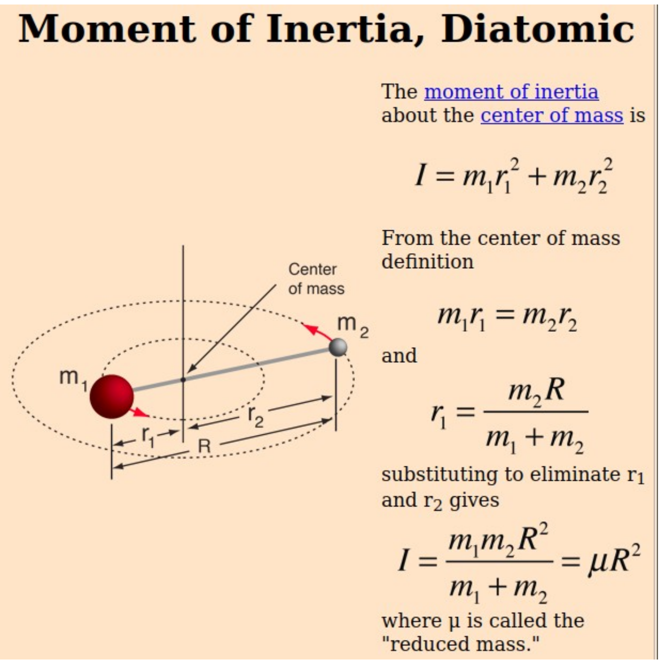
where $\vec{\Omega}$ is the angular velocity of the earth's frame relative to an inertial frame (for example far off stars) and ω_0 is the natural frequency of the pendulum.
By defining a complex number $\eta = x + iy$, solve the coupled differential equation and show that the general solution of motion is

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right) \qquad (2)$$

If at $t = 0$ the pendulum is at rest with $x = A$ and $y = 0$, find the coefficient C_1 and C_2, and show that because $\Omega << \omega_0$ they are well approximated as $C_1 = C_2 = A/2$ giving the solution

$$\eta(t) = x(t) + iy(t) = Ae^{-i\Omega_z t}\cos\omega_0 t \qquad (3)$$

Moment of Inertia of a Diatomic molecule



Moment of Inertia, Diatomic

The moment of inertia about the center of mass is

$$I = m_1 r_1^2 + m_2 r_2^2$$

From the center of mass definition

$$m_1 r_1 = m_2 r_2$$

and

$$r_1 = \frac{m_2 R}{m_1 + m_2}$$

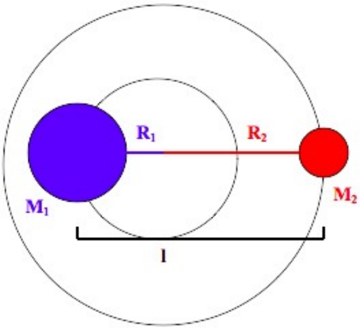
substituting to eliminate r_1 and r_2 gives

$$I = \frac{m_1 m_2 R^2}{m_1 + m_2} = \mu R^2$$

where μ is called the "reduced mass."

Rotational Energy states of diatomic molecules

The rotations of a diatomic molecule can be modelled as a rigid rotor. This rigid rotor model has two masses attached to each other with a fixed distance between the two masses.



It has an inertia (I) that is equal to the square of the fixed distance between the two masses multiplied by the reduced mass of the rigid rotor.

$$I_e = \mu r_e^2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Using quantum mechanical calculations it can be shown that the energy levels of the rigid rotator depend on the inertia of the rigid rotator and the quantum rotational number J[2].

$$E(J) = B_e J(J+1)$$

$$B_e = \frac{h}{8\pi^2 c I_e}$$

However, this rigid rotor model fails to take into account that bonds do not act like a rod with a fixed distance, but like a spring. This means that as the angular velocity of the molecule increases so does the distance between the atoms. This leads us to the nonrigid rotor model in which a centrifugal distortion term (De) is added to the energy equation to account for this stretching during rotation.

$$E(J)(cm^{-1}) = B_e J(J+1) - D_e J^2(J+1)^2$$

This means that for a diatomic molecule the transitional energy between two rotational states equals

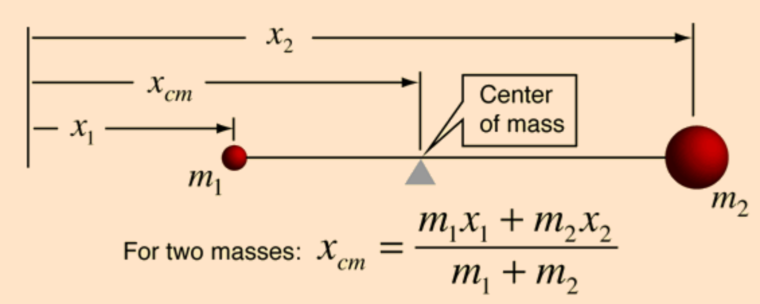
$$E = B_e[J'(J'+1) - J''(J''+1)] - D_e[J'^2(J'+1)^2 - J''^2(J'+1)^2]$$

Where J' is the quantum number of the final rotational energy state and J'' is the quantum number of the initial rotational energy state. Using the selection rule of $\Delta J = \pm 1$ the spacing between peaks in the microwave absorption spectrum of a diatomic molecule will equal

$$E_R = (2B_e - 4D_e) + (2B_e - 12D_e)J'' - 4D_e J''^3$$

Centre of Mass

The terms "centre of mass" and "centre of gravity" are used synonymously in a uniform gravity field to represent the unique point in an object or system which can be used to describe the system's response to external forces and torques. The concept of the centre of mass is that of an average of the masses factored by their distances from a reference point. In one plane, that is like the balancing of a seesaw about a pivot point with respect to the torques produced.



If you are making measurements from the center of mass point for a two-mass system then the centre of mass condition can be expressed as

$$m_1 r_1 = m_2 r_2 \qquad m_1 = m_2 \frac{r_2}{r_1}$$

where r_1 and r_2 locate the masses. The centre of mass lies on the line connecting the two masses.

The impulse I of a constant force F acting for a time t is defined as $F \times t$.

$$I = F \times t.$$

By Newton's second law, $F = ma$.

If u and v are the initial and final velocities of the particle,

$$a = (v - u)/t$$

$$\therefore \qquad I = Ft = mat = m\left(\frac{v-u}{t}\right)t = m\,(v-u)$$

Thus *the impulse of a force is equal to the change in momentum produced.*

Impulsive Force : Definition. *An impulsive force is an infinitely great force acting for a very short interval of time, such that their product is finite.*

The force and the time cannot be measured because one is too great and the other is too small. Nevertheless, their product, which is definite, is capable of measurement. This we have seen, is the impulse of the impulsive force and is equal to the change in momentum produced. Hence an impulsive force is always measured by the change in momentum produced. In practice, the conditions of an impulsive force are never realized. Some approximate examples of impulsive force are : (1) the blow of a hammer on a pile and (2) the force exerted by the bat on a cricket ball.

Collisions

Elastic and Inelastic collisions : There are two types of collision :

(*i*) elastic and (*ii*) inelastic.

(*i*) Elastic collisions are those in which the total kinetic energy before and after the collision remains unchanged. Collisions between atomic, nuclear and fundamental particles are the true elastic collisions. Collisions between ivory or glass balls can be treated as approximately elastic collisions. In such a collision between two particles, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

and
$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

where m_1 and m_2 are the respective masses of the two particles and u_1, u_2 and v_1, v_2 their velocities *before* and *after* the collision.

(*ii*) If the K. E. is not conserved, the collision is said to be inelastic. When two bodies stick together after collision, the collision is said to be completely inelastic. For example, the collision between a bullet and its target is completely inelastic when the bullet remains embedded in the target.

Completely Inelastic Collision : Suppose a body of mass m_1 moving with a velocity u_1 collides with a body of mass m_2 moving with velocity u_2 in the same direction. The two bodies stick together after collision and they move with a final common velocity V in the same direction as the original. It is not necessary to restrict the discussion to one dimensional motion. Using only the conservation of momentum principle,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\,V.$$

From this the value of V can be determined if u_1 and u_2 are known.

Here, $(u_1 - u_2)$ and $(v_1 - v_2)$ are their relative velocities, before and after the impact. e lies between 0 and 1. If $e = 0$, the bodies are called perfectly plastic bodies. If $e = 1$, the bodies are called perfectly elastic bodies. For two glass balls, $e = 0.94$; For two lead balls, $e = 0.2$.

Definition of coefficient of restitution. *The ratio, with a negative sign, of the relative velocity of two bodies after impact to their relative velocity before impact is called the coefficient of restitution.*

2. Motion of two smooth bodies perpendicular to the line of impact. When two smooth bodies impinge, there is no tangential action between them. Hence there is no change of momentum along the common tangent. Hence, there is no change of velocity for either body along the tangent. In other words, there is no change in the velocity of a body in a direction perpendicular to the common normal due to impact.

3. Principle of conservation of momentum. The total momentum of two bodies after impact along the *common normal* should be equal to the total momentum before the impact along the same direction.

The above three principles are sufficient to determine the change in motion of two impinging smooth bodies.

Definitions. (*i*) Two bodies are said to impinge *directly* when the direction of motion of each is along the common normal at the point where they touch.

(*ii*) Two bodies are said to impinge *obliquely* if the direction of motion of either or both is not along the common normal at the point of contact.

(*iii*) The common normal at the point of contact is called the line of impact. Thus, in the case of two spheres the line of impact is the line joining their centres.

Collision: Direct impact of two smooth spheres (Determination of final velocities)

A smooth sphere of mass m_1 moving with a velocity u_1 impinges on another smooth sphere of mass m_2 moving in the same direction with velocity u_2. If e is the coefficient of restitution between them, find the velocities of the spheres after impact.

Since the spheres are smooth, there is no impulsive force on either along the common tangent. Hence in this direction their velocities after impact are the same as their original velocities *i.e.*, zeroes.

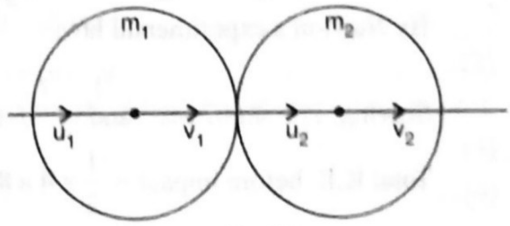


Fig. 8.2

Let v_1 and v_2 be the velocities of the two spheres along the common normal after impact [Fig. 8.2].

By the principle of conservation of momentum,

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \qquad \text{....(1)}$$

By Newton's experimental low,

$$v_1 - v_2 = -e\,(u_1 - u_2) \qquad \text{...(2)}$$

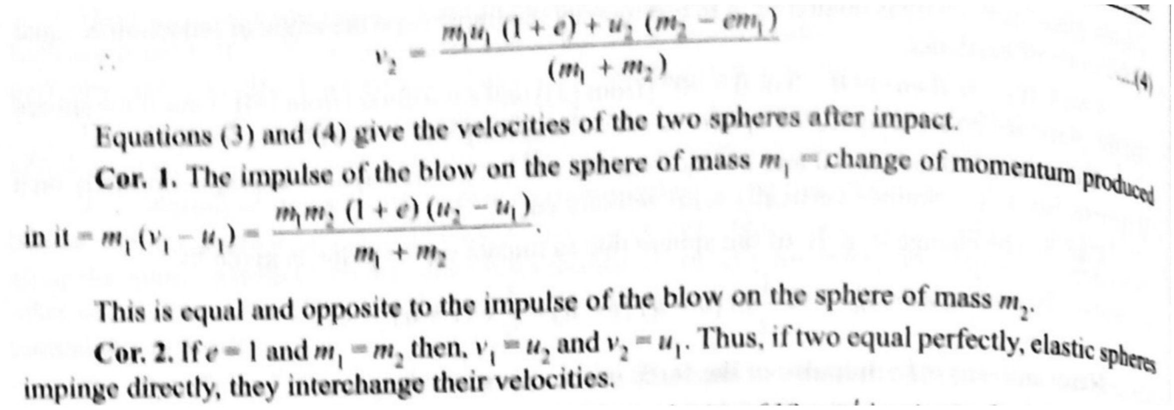
Multiplying (2) by m_2 and adding to (1),

$$v_1\,(m_1 + m_2) = m_2 u_2\,(1 + e) + u_1\,(m_1 - e m_2)$$

$$\therefore \quad v_1 = \frac{m_2 u_2\,(1 + e) + u_1\,(m_1 - e m_2)}{(m_1 + m_2)} \qquad \text{...(3)}$$

Multiplying (2) by m_1 and subtracting from (1),

$$v_2\,(m_1 + m_2) = m_1 u_1\,(1 + e) + u_2\,(m_2 - e m_1)$$



$$\therefore \quad v_2 = \frac{m_1u_1(1+e) + u_2(m_2 - em_1)}{(m_1 + m_2)} \qquad ...(4)$$

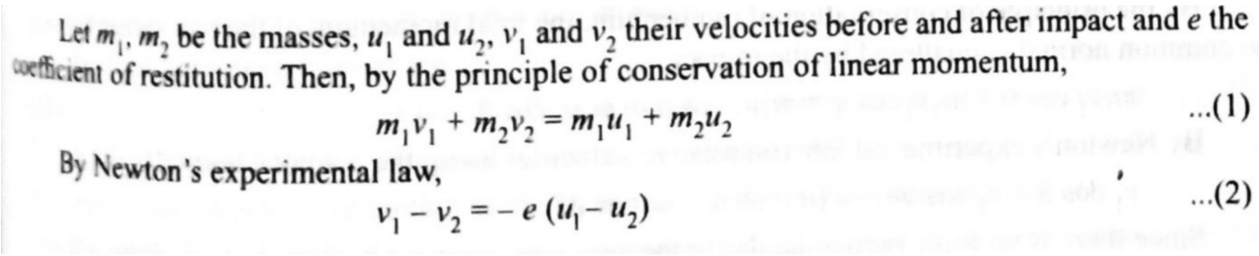
Equations (3) and (4) give the velocities of the two spheres after impact.

Cor. 1. The impulse of the blow on the sphere of mass m_1 = change of momentum produced in it $= m_1(v_1 - u_1) = \dfrac{m_1m_2(1+e)(u_2 - u_1)}{m_1 + m_2}$.

This is equal and opposite to the impulse of the blow on the sphere of mass m_2.

Cor. 2. If $e = 1$ and $m_1 = m_2$ then, $v_1 = u_2$ and $v_2 = u_1$. Thus, if two equal perfectly, elastic spheres impinge directly, they interchange their velocities.

Determination of Loss of kinetic energy

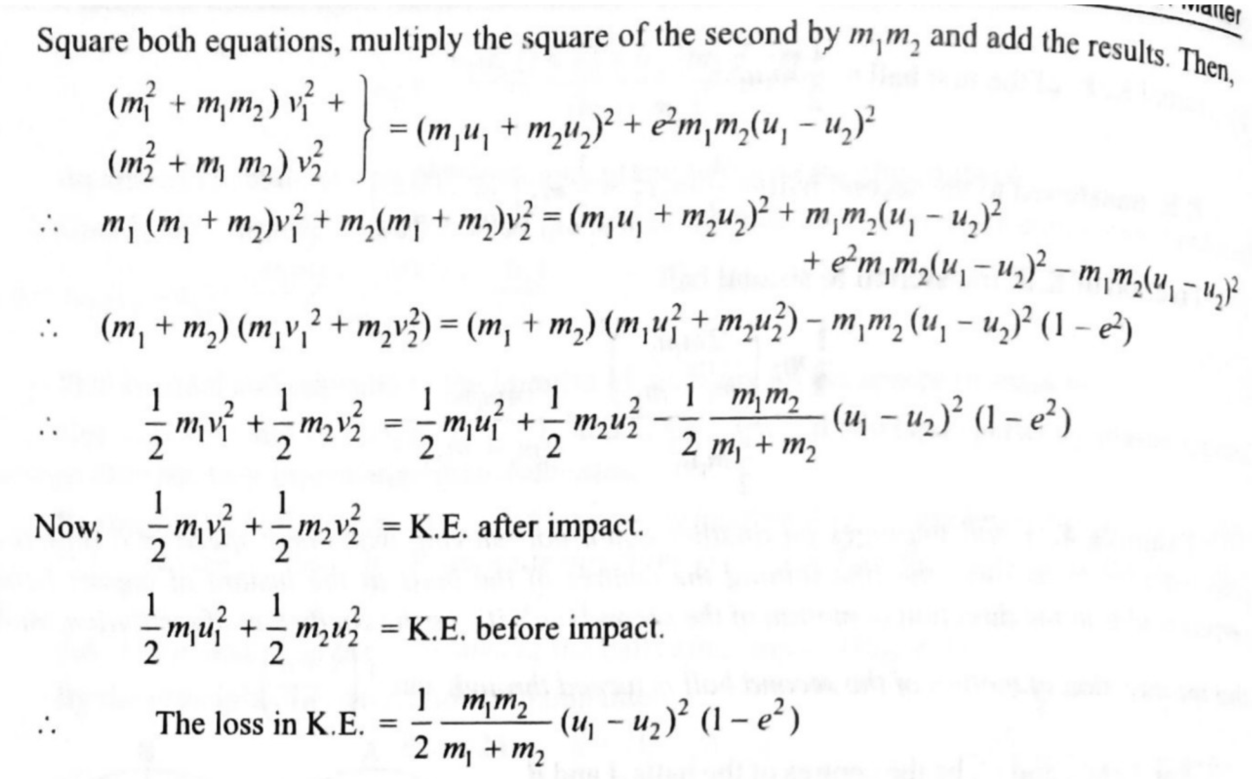


Let m_1, m_2 be the masses, u_1 and u_2, v_1 and v_2 their velocities before and after impact and e the coefficient of restitution. Then, by the principle of conservation of linear momentum,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \qquad ...(1)$$

By Newton's experimental law,

$$v_1 - v_2 = -e(u_1 - u_2) \qquad ...(2)$$



Square both equations, multiply the square of the second by m_1m_2 and add the results. Then,

$$\left.\begin{array}{l}(m_1^2 + m_1m_2)v_1^2 + \\ (m_2^2 + m_1m_2)v_2^2\end{array}\right\} = (m_1u_1 + m_2u_2)^2 + e^2m_1m_2(u_1 - u_2)^2$$

$$\therefore \quad m_1(m_1 + m_2)v_1^2 + m_2(m_1 + m_2)v_2^2 = (m_1u_1 + m_2u_2)^2 + m_1m_2(u_1 - u_2)^2 + e^2m_1m_2(u_1 - u_2)^2 - m_1m_2(u_1 - u_2)^2$$

$$\therefore \quad (m_1 + m_2)(m_1v_1^2 + m_2v_2^2) = (m_1 + m_2)(m_1u_1^2 + m_2u_2^2) - m_1m_2(u_1 - u_2)^2(1 - e^2)$$

$$\therefore \quad \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2)$$

Now, $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$ = K.E. after impact.

$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$ = K.E. before impact.

$$\therefore \quad \text{The loss in K.E.} = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(u_1 - u_2)^2(1 - e^2)$$

Note : When $e = 1$, the loss of K. E. is zero. In general $e < 1$ so that $(1 - e^2)$ is positive. $(u_1 - u_2)^2$ is always positive. Hence, there is always a loss of K.E. due to impact. The K.E. lost during impact is converted into (*i*) sound, (*ii*) heat or (*iii*) vibration or rotation of the colliding bodies.

When $e = 0$, the loss in K.E. $= \dfrac{1}{2} \dfrac{m_1 m_2 (u_1 - u_2)^2}{(m_1 - m_2)}$

i.e., there is maximum loss of K.E. on impact of plastic bodies.