Notes of Mechanics
Unit-I: Laws of Motion
B. Sc Physics Semester I
As per Pondicherry University Syllabus
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Scalar and vector quantities

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Physical quantities may be divided into two main classes.

- ➔ **Scalar quantities or scalars** : A scalar quantities or scalars are those physical quantities which possess only magnitude or numerical value. For example : volume, mass, temperature, time and speed etc. They are subjected to the ordinary algebraical laws of addition and multiplication.

- ➔ The algebric laws of addition are

i. Laws of Commutation :

According to this law, the result of addition or multiplication of a number of scalars is quite independent of the order in which they may be taken.

(A + B) = (B + A) and **(A×B) = (B×A)**

ii. Laws of association :

According this law, the sum or the product of a finite number of scalars in quite independent of the manner in which they may be grouped or associated.

Thus

A + B + C = (A+B) + C = A + (B+C) = (A+C) + B
A×(B×C)= (A×B)×C = (A×C)×B

iii. Laws of distribution :

According to this law, in expressions involving both addition and multiplication, the result in the same as the sum of the individual term wise products. For example

A×(B+C) = (A×B) + (A×C) and **(A+B)×C = (A×C) + (B×C)**
iv.

- ➔ **Vector quantities or vector** : Vector quantities or vectors are those quantities which possess magnitude as well as directions. For example : force, velocity, momentum, weight etc. They are subjected to parallelogram law of addition and vector multiplication (scalar and vector product).
  Therefore, to qualify as a vector, a physical quantity must not only possess magnitude and direction but also must satisfy the parallelogram law of vector addition.

Scalar and vector products

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Unit vector : A vector of unit magnitude is called a unit vector and the notation for it in the direction of $\vec{A}$, is denoted by $\hat{A}$.

$$\vec{A}=\hat{A}\,A$$

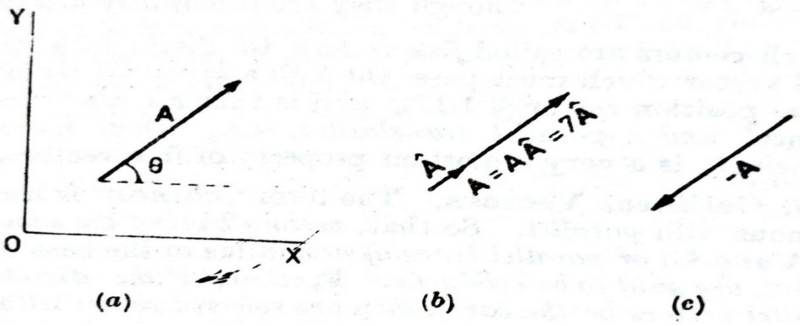
A unit vector simply indicates the direction of the vector along with which it is associated.

Zero Vector : A vector with zero magnitude is called a zero or a null vector. All the zero or null vectors are considered to be equal and their directions are quite arbitrary.

Vectors other than null or zero vectors are referred to as ***proper vectors.***

Graphical representation of a vector

Graphically, a vector is represented by an arrow drawn to a chosen scale, parallel to the direction of the vector. The length and direction of the arrow represents the magnitude and direction of the vector respectively.



Multiplication and Division of Vector by a scalar

$$(\vec{A}+\vec{A}+\vec{A}+\text{.............} \textit{upto mth terms})=m\vec{A}$$

$$(m+n)\vec{A}=m\vec{A}+n\vec{A}$$
$$m(n\vec{A})=n(m\vec{A})=(mn)\vec{A}$$

where m and n are constants or scalar quantity.

Similarly, the division of a vector $\vec{A}$ by a non-zero scalar m is defined as the multiplication of a vector $\vec{A}$ by $\frac{1}{m}$.

$$\frac{\vec{A}}{m}=\frac{1}{m}(\vec{A})$$

Equality of two vectors

All vectors with the same magnitude and direction, are said to be equal despite their entirely different locations in space and remain so even if moved parallel to themselves.

Addition and subtraction of two vectors

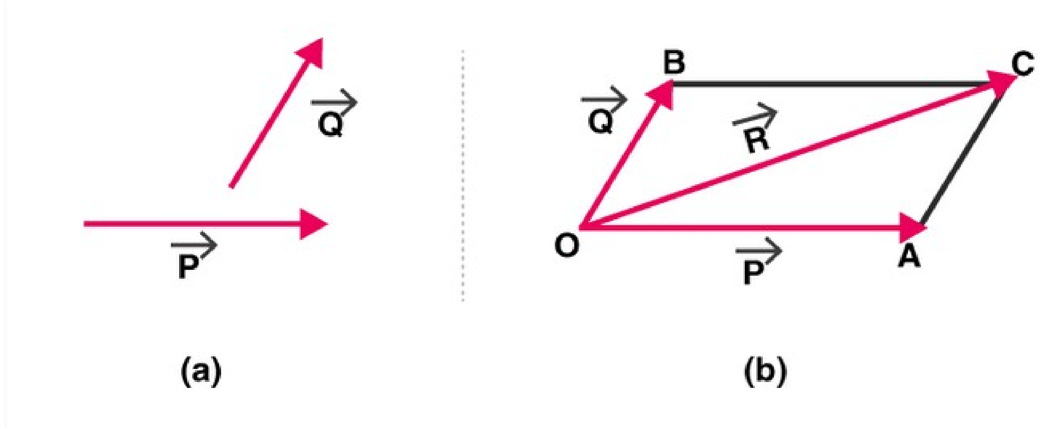
- **Addition :**

(a) Parallelogram law of vector addition

If two vectors are acting simultaneously at a point, then it can be represented both in magnitude and direction by the adjacent sides drawn from a point. Therefore, the resultant vector is completely represented both in direction and magnitude by the diagonal of the parallelogram passing through the point.

$$\vec{O}A+\vec{O}B = \vec{O}C$$

$$\vec{P}+\vec{Q}=\vec{R}$$



Parallelogram Law of Addition of Vectors Procedure

The steps for the parallelogram law of addition of vectors are:

1. Draw a vector using a suitable scale in the direction of the vector
2. Draw the second vector using the same scale from the tail of the first vector
3. Treat these vectors as the adjacent sides and complete the parallelogram
4. Now, the diagonal represents the resultant vector in both magnitude and direction

(b) **The triangle Law of vector addition**

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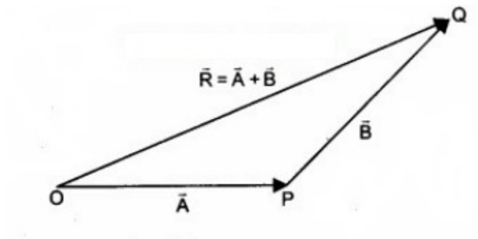
The triangle law of vector addition follows from the parallelogram law and states that if the tail end of one vector be placed at the head or the arrow end of another vector, their sum or resultant $\vec{R}$ is drawn from the tail end of the first vector to the head end of the other vector.

The resultant $\vec{R}$ is the same irrespective of the order in which the vectors $\vec{A}$ and $\vec{B}$ are taken.

So, we have $$\vec{R}=\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

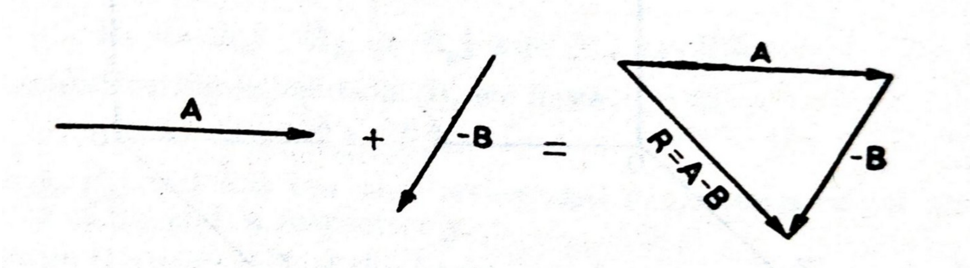


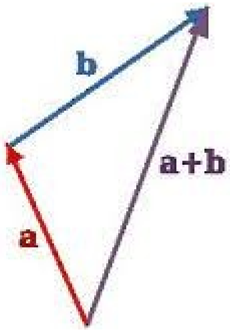
Triangle law of addition

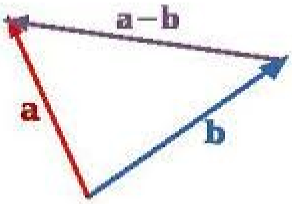
It is to be noted that vectors need not lie in the same plane for the laws of vector addition to be applicable.

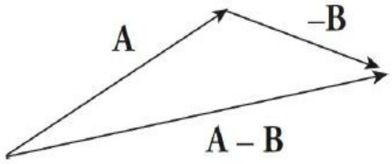
➢ Subtraction :

If the sum of two vectors $\vec{A}$ and $\vec{B}$ is equal to a vector of zero magnitude, we have $\vec{A}+\vec{B}=0$









Rectangular Components of a vector

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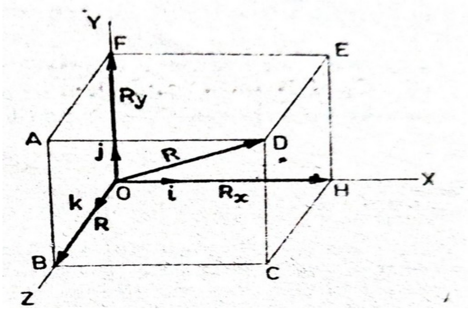
With O as the origin, let us consider that the vector $\vec{OD}$ is $\vec{R}$. Then on OD as the body diagonal, let us construct a rectangular parallelopiped with its three edges along the three coordinate axes, as shown in the figure. Let us consider that the vector intercepts of the three components of $\vec{R}$ along the three axes respectively be R_x, R_y and R_z. Then, we have



$$\vec{R}=\vec{R}_x+\vec{R}_y+\vec{R}_z \quad \text{……………………...(1)}$$

Let $\hat{i},\hat{j}$ and $\hat{k}$ be the unit vectors along the three coordinate axes, chosen as the base vector or forming what is called the orthogonal traid of vectors. Then, we have

$$\vec{R}=R_x\hat{i}+R_y\hat{j}+R_z\hat{k} \quad \text{…………………….(2)}$$

Here $R_x\hat{i}, R_y\hat{j}$ and $R_z\hat{k}$ are the orthogonal projections of $\vec{R}$ on the directions of $\hat{i},\hat{j}$ and $\hat{k}$ respectively.

From the geometry of the figure, we have

$$OD^2=OH^2+OF^2+OB^2$$

$$R^2=R_x^2+R_y^2+R_z^2 \quad \text{……………………..(3)}$$

where ***R*** is the modulus or magnitude of the vector $\vec{R}$.

Thus, the square of the magnitude or modulus of a vector is equal to the sum of the squares of its rectangular components.

Therefore $$R=\sqrt{(R_x^2+R_y^2+R_z^2)} \quad \text{……………………………..(4)}$$

Clearly, $R_x=R\cos\alpha$, where α is the angle between $\vec{R}$ and $\vec{R}_x$.

$$\cos\alpha=\frac{R_x}{R} \quad \text{……………………..(4a)}$$

Similarly $$\cos\beta=\frac{R_y}{R} \quad \text{………..(4b)} \quad \text{and} \quad \cos\gamma=\frac{R_z}{R} \quad \text{……………….(4c)}$$

Here, (4a), (4b) and (4c) represents the ***direction cosines*** of the vector $\vec{R}$ since they help in determining the direction of the vector.

Now, dividing the equation (3) by R^2 , we have

$$1=\frac{R_x^2}{R^2}+\frac{R_y^2}{R^2}+\frac{R_z^2}{R^2} \quad \text{………………………...(5)}$$

Using (4a), (4b) and (4c) in (5) we have

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \quad \text{……………………..(6)}$$

$$l^2+m^2+n^2=1 \quad \text{………………………..(7)}$$

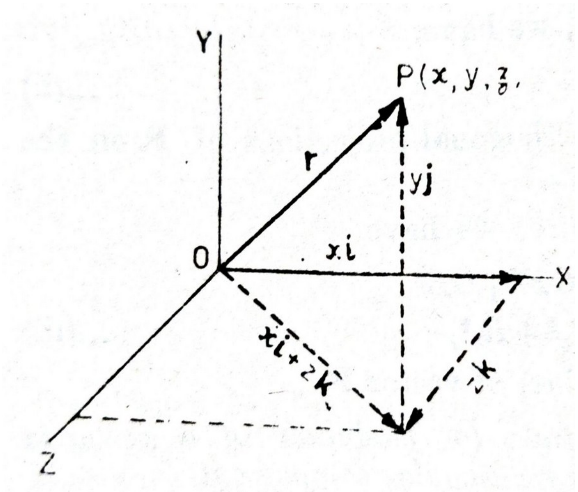
That is, the sum of the squares of the three direction cosines of a vector is equal to unity.

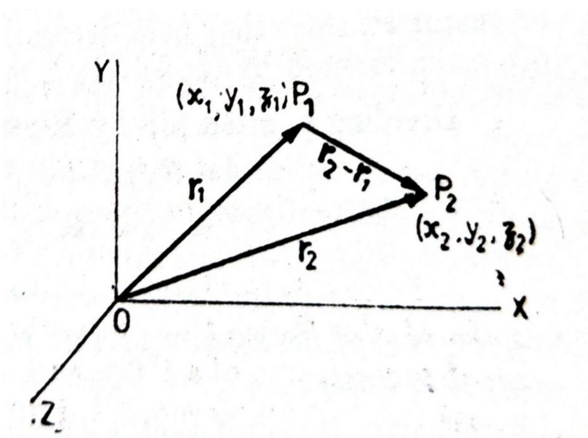
Again dividing equation (2) by R we have

$$\frac{\vec{R}}{R}=\frac{R_x}{R}\hat{i}+\frac{R_y}{R}\hat{j}+\frac{R_z}{R}\hat{k}$$

$$\textit{unit vector of } \vec{R}=(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k} \quad \text{…………………...(8)}$$

Position Vector





The position vector of a point ***P*** from any assigned point, such as the origin ***O*** of the Cartesian coordinate system is uniquely specified by the vector $\vec{OP}=\vec{r}$, is called the position vector of the point ***P*** with respect to ***O***.

The coordinates of P being (x, y, z), we have

………………………..(1)

where $r=|\vec{r}|=\sqrt{(x^2+y^2+z^2)}$.

Now, let us consider two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ as shown in the figure, with $\vec{OP}_1=\vec{r}_1$ and $\vec{OP}_2=\vec{r}_2$ as their respective position vectors.

Therefore, we can write

$$\vec{r}_1=x_1\hat{i}+y_1\hat{j}+z_1\hat{k} \qquad \text{and} \qquad \vec{r}_2=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}$$

Therefore, $\vec{P_1P_2}=\vec{r}_2-\vec{r}_1=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$ ………………...(2)

Condition for coplanarity of Vectors

If $\vec{A}$ and $\vec{B}$ be two vectors in the same plane, the vector $\vec{R}=m_1\vec{A}+m_2\vec{B}$ will also be coplanar with $\vec{A}$ and $\vec{B}$, irrespective of the value of the scalars m_1 and m_2 .

In case of three or more vectors , they are said to be coplanar when they are parallel to the same plane.

Let us assume that $\vec{A}$, $\vec{B}$ and $\vec{C}$ be three coplanar vectors. Then,

$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$$

$$\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$$

$$\vec{C}=C_x\hat{i}+C_y\hat{j}+C_z\hat{k}$$

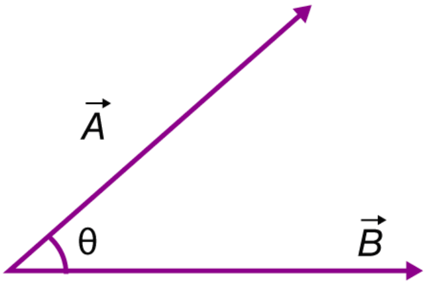
For coplanarity, the condition is

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = 0$$

Product of two vectors

i. **Scalar Product :**

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}.\vec{B}$ and it is read as $\vec{A}$ dot $\vec{B}$. It is also known as ***dot product*** of two vectors. It is sometimes called ***direct product*** of two vectors.



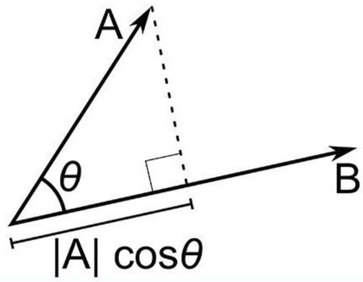
It is defined as the product of the magnitudes of the two vectors $\vec{A}$ ***and*** $\vec{B}$ ***, and the cosine of their included angle*** θ ***as shown in the figure.***

Thus $\vec{A}.\vec{B}$ = $|\vec{A}||\vec{B}|\cos\theta = AB\cos\theta = AB\cos(\vec{A},\vec{B})$

Since $\cos(\vec{A},\vec{B})=\cos(\vec{B},\vec{A})$ **,** the scalar product is clearly commutative and hence

$$\vec{A}.\vec{B}=\vec{B}.\vec{A}$$

Further, since $|\vec{A}|\cos\theta$ = $A\cos\theta$ is the projection or resolute of $\vec{A}$ in the direction of the $\vec{B}$.



Similarly $|\vec{B}|\cos\theta$ = $B\cos\theta$ is the projection or resolute of $\vec{B}$ in the direction of the $\vec{A}$.

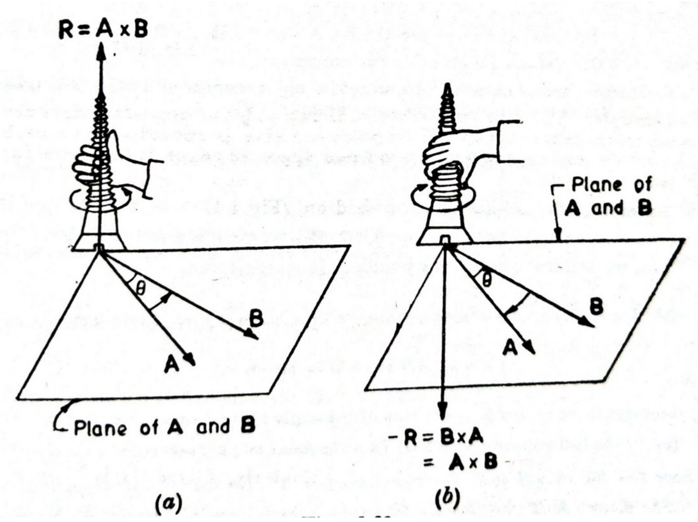
$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta = A(B\cos\theta) = BA(\cos\theta)$$

Therefore, the scalar product of two vectors ($\vec{A}$ ***and*** $\vec{B}$ ***) is the product of the magnitudes of either vector and the projection of the other vector in its direction.***

ii. Vector Product :

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$ and it is read as $\vec{A}$ cross $\vec{B}$. It is also known as ***cross product*** of two vectors. It is sometimes called ***outer product*** of two vectors.

It is defined as the product of the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$, and the sine of their included angle θ as shown in the figure.



Thus $\vec{A}\times\vec{B}$ = $|\vec{A}||\vec{B}|\sin\theta\hat{n}=A\,B\sin\theta\hat{n}=A\,B\sin(\vec{A},\vec{B})\hat{n}$

where $0\ll\theta\ll180^o$ and $\hat{n}$ is the unit vector which gives the direction of the vector $\vec{R}$. The unit vector $\hat{n}$ is perpendicular to the plane containing both $\vec{A}$ and $\vec{B}$, as shown in the figure.

Torque or moment of a force

Let us consider that a force $\vec{F}$ be acting on a body free to rotate about O as shown in the figure. Let $\vec{r}$ be the position vector of any point P on the line of action of the force. Then we know that

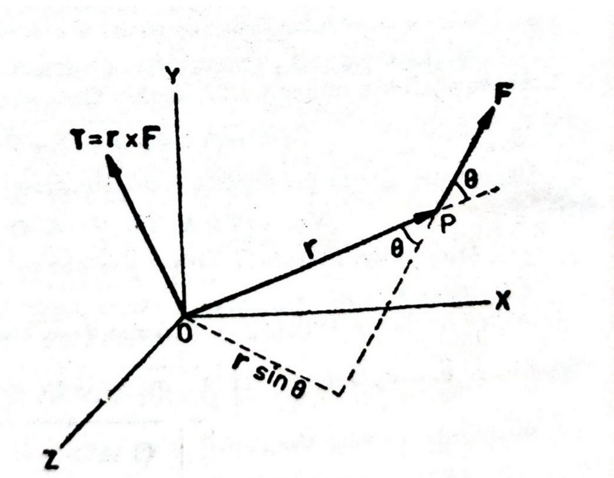
Torque = (Force) X (perpendicular distance of its line of action from O)

Therefore, we can write

$$\tau=F\,r\sin\theta$$

Further modifying we have

$$\vec{\tau}=|\vec{F}||\vec{r}|\sin\theta=(\vec{r}\times\vec{F})$$



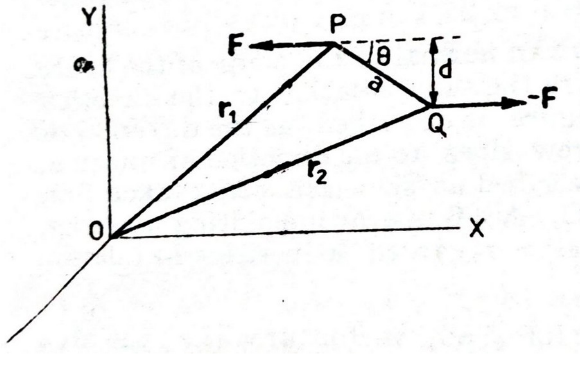
Direction of the torque $\vec{\tau}$ is perpendicular to the plane containing $\vec{F}$ and $\vec{r}$.

Couple

A couple is a combination of two equal , opposite and parallel forces. Let $\vec{F}$ and $-\vec{F}$ be two such forces acting at points P and Q as shown in the figure. The position vectors of the points P and Q are $\vec{r}_1$ and $\vec{r}_2$ respectively.

Then, since the moment of the couple $\vec{C}$ with respect to O is equal to the sum of the moments with respect to O, of the two forces constituting the couple, we have

$$\vec{C}=(\vec{r}_1\times\vec{F})+(\vec{r}_2\times(-\vec{F}))=(\vec{r}_1-\vec{r}_2)\times\vec{F}$$

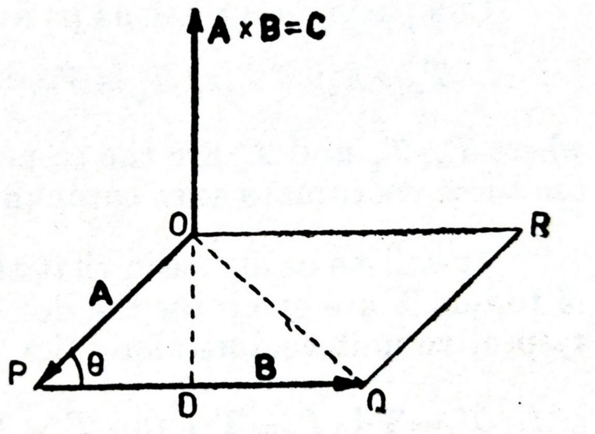


Since $(\vec{r}_1-\vec{r}_2)=\vec{a}$, where $\vec{a}$ lies in the same plane with $\vec{F}$.

Therefore, we have $\vec{C}=(\vec{a}\times\vec{F})$

Area of a parallelogram

Let us consider that the vectors $\vec{A}$ and $\vec{B}$ form the adjacent sides of a parallelogram OPQR, inclined to each other at an angle θ . Then, if OD be the perpendicular dropped from O on to PQ, we have area of the parallelogram



= PQ (OD)=B (Asin θ)
=AB sin θ =AB sin (A, B)

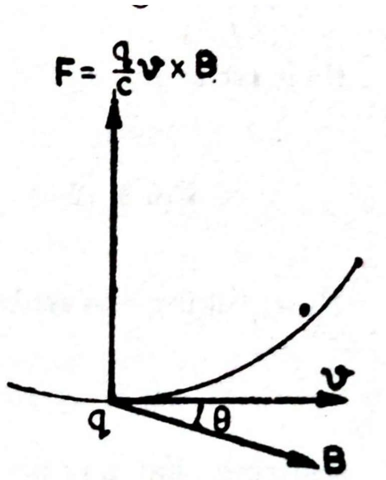
which is obviously twice the area of the triangle OPQ with the same adjacent sides A and B.

Clearly **AB sin** θ **=AB sin (A, B)** is the magnitude of the vector product $\vec{A}\times\vec{B}=\vec{C}$, say, whose direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, i.e., to the plane of the parallelogram.

Now, an area, by itself, has no sign but may be regarded positive or negative in relation to the direction in which its boundary is described. $\vec{A}\times\vec{B}=\vec{C}$, therefore, represents a vector area which gives both the magnitude and the orientation of the area of the parallelogram.

Force on a moving charge in a magnetic field

Imagine a charge q to be moving with velocity $\vec{v}$ at an angle θ with a magnetic field $\vec{B}$ at any given instant as shown in the figure. Then, the force acting on it in a direction perpendicular to $\vec{B}$ as well as $\vec{v}$ is F = qvB sinθ, where F, v and B are the magnitudes of the force, velocity and the magnetic field respectively. In vector form, therefore, we may put it as



$$\vec{F}=q(\vec{v}\times\vec{B})$$

This is called ***Lorentz force law***, with the force itself referred to as the ***Lorentz force***.

In most cases, q is taken in esu and B in emu (i.e., gauss). Since 1 esu charge=(1/c) emu, where c is the velocity of light in vacuum, we have q esu = (q/c) emu and, therefore,

$$\vec{F}=\frac{q}{c}(\vec{v}\times\vec{B})$$

In case the charge also simultaneously passes through an electric field $\vec{E}$, an additional force $q\vec{E}$ acts upon it and the Lorentz force law then takes the form

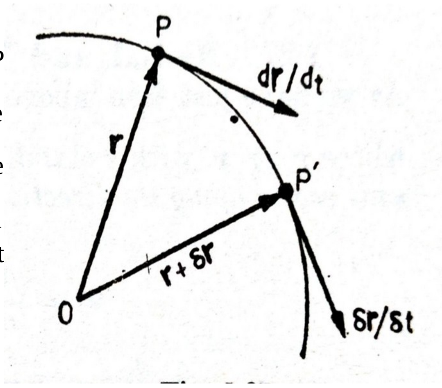
$$\vec{F}=q\vec{E}+\frac{q}{c}(\vec{v}\times\vec{B})$$

Vector derivatives (velocity and acceleration for linear motion)

Let us consider that $\vec{r}$ be a *single-valued function* of a scalar variable ***t*** such that for every value of t there exists only one value of $\vec{r}$. Then, as t varies continuously, $\vec{r}$ also does so.

We are particularly interested in the case in which represents the time variable and $\vec{r}$ stands for the position vector of a moving particle with respect to a fixed origin O. Then, as f varies continuously, the point moves along a continuous curve in space. So that, if $\vec{r}$ and $(\vec{r}+\delta\vec{r})$ be the position vectors of the point in positions P and P' relative to origin O for the values t and $(t+\delta t)$ of the scalar variable, we have

Change in the value of $\vec{r}$ = $\delta\vec{r}$



The quotient $\frac{\delta\vec{r}}{\delta t}$ is also a vector. As $\delta t\to 0$, the point P' approaches P and the chord PP' tends to coincide with the tangent to the curve at P. The limiting value of $\frac{\delta\vec{r}}{\delta t}$ as $\delta t\to 0$ is $\frac{d\vec{r}}{dt}$ and is a vector those direction is that of the tangent at P in the sense in which increases. It is called the time derivative of $\vec{r}$ or the differential coefficient of $\vec{r}$ with respect to t.

We thus have

$$\frac{d\vec{r}}{dt}=\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}$$

When this limit exists, the function $\vec{r}$ is said to be differentiable. The second and third derivatives of $\vec{r}$ are respectively $\frac{d^2\vec{r}}{dt^2}$ and $\frac{d^3\vec{r}}{dt^3}$.

Clearly, $\delta\vec{r}$ represents the displacement of the particle in time interval δt and, therefore, $\frac{\delta r}{\delta t}$ gives its average velocity during interval δt . The limiting value of this average velocity, as $\delta t\to 0$, is the instantaneous velocity $\vec{v}$ of the particle. Thus, we have

$$v=\frac{d\vec{r}}{dt}$$

along the tangent to the path of the particle.

Proceeding in the same manner if, $\delta\vec{v}$ be the increase in the velocity $\vec{v}$ of the particle during the time-interval δt , the rate of change of velocity or the average acceleration during the interval is $\frac{\delta\vec{v}}{\delta t}$ and therefore, instantaneous acceleration a of the particle is the limiting value $\frac{d\vec{v}}{dt}$ of $\frac{\delta\vec{v}}{\delta t}$ as $\delta t\to 0$.

Thus,

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{r}}{dt^2}$$

Now, since $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and since x, y and z are functions of time, we also have

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}$$

and

$$\vec{a}=(\frac{d\vec{v}}{dt})=(\frac{d^2\vec{r}}{dt^2})=(\frac{d^2x}{dt^2})\hat{i}+(\frac{d^2y}{dt^2})\hat{j}+(\frac{d^2z}{dt^2})\hat{k}$$

Radial and Transverse Components of Velocity

As we have just seen above, $\vec{v}=\frac{d\vec{r}}{dt}=\frac{dx}{dt}\hat{i}+\frac{dy}{dt}\hat{j}+\frac{dz}{dt}\hat{k}$

where $\vec{r}=r\hat{r}$, with r standing for the magnitude of $\vec{r}$ and $\hat{r}$ for the unit vector along its direction. We, therefore, have

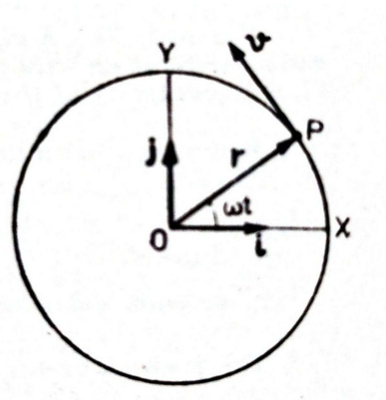
$$\vec{v}=\frac{d\vec{r}}{dt}=\left(\frac{dr}{dt}\right)\hat{r}+r\left(\frac{d\hat{r}}{dt}\right)$$

The vector $\frac{dr}{dt}\hat{r}$ is called the radial component of velocity $\vec{v}$ of the particle, its magnitude being obviously $\frac{dr}{dt}$.

And the vector $r\frac{d\hat{r}}{dt}$ is called the transverse component of velocity $\vec{v}$ of the particle, because it is perpendicular to $\vec{r}$.

Vector derivatives (velocity and acceleration for circular motion)

Let us try to obtain expressions for the velocity and acceleration of a particle P moving at a constant speed along a circular path in the zy plane, say, of constant radius r as shown in the figure.



The position vector r, a function of the scalar variable t, moving with constant angular velocity (or angular frequency) of magnitude o may, at any given instant t, be expressed in terms of its components along the axes of z and y. Thus, if i and j be the unit vectors along the two axes respectively, we have

$$\vec{r}=r\left(\cos\omega t\,\hat{i}+\sin\omega t\,\hat{j}\right)$$

The velocity $\vec{v}$ of particle P is thus given by

$$\vec{v}=\frac{d\vec{r}}{dt}=r\frac{d}{dt}\left(\cos\omega t\,\hat{i}+\sin\omega t\,\hat{j}\right)$$

where r is constant for circular motion.
Therefore

$$\vec{v}=r\left(-\omega\sin\omega t\,\hat{i}+\omega\cos\omega t\,\hat{j}\right)$$

$$\Rightarrow\vec{v}=r\left(-\omega\sin\omega t\,\hat{i}+\omega\cos\omega t\,\hat{j}\right)=r\,\omega\left(-\sin\omega t\,\hat{i}+\cos\omega t\,\hat{j}\right)$$

We know that

$$v^2=\vec{v}\,.\,\vec{v}=\omega^2r^2\left(\sin^2\omega t+\cos^2\omega t\right)=\omega^2r^2$$

which gives $v=\omega r$

Similarly, the acceleration is given by

$$\vec{a}=\frac{d\vec{v}}{dt}=r\,\omega(-\omega\cos\omega t\,\hat{i}-\omega\sin\omega t\,\hat{j})=-\omega^2 r(\cos\omega t\,\hat{i}+\sin\omega t\,\hat{j})=-\omega^2\vec{r}$$

Angular velocity vector

The angular velocity of a particle, having magnitude as well as direction, is obviously a vector. To obtain its value, we note that the linear velocity $\vec{v}$ of the particle is, at any instant, perpendicular to both the angular velocity $\vec{\omega}$ and the radius vector $\vec{r}$. So that, the vector equation corresponding to the relation $v=\omega r$ comes out to be $\vec{v}=\omega\times\vec{r}$, showing that the linear velocity of the particle is the cross product of the angular velocity vector and the position vector with respect to a fixed point on the axis of rotation. The cross product of this with $\vec{r}$ gives

$$\vec{r}\times\vec{v}=\vec{r}\times(\vec{\omega}\times\vec{r})=\omega(\vec{r}.\vec{r})-\vec{r}(\vec{r}.\vec{\omega})=r^2\omega$$

(because $\vec{r}.\vec{\omega}=0$, $\vec{r}$ and $\vec{\omega}$ being perpendicular to each other),

Derivatives of a vector with respect to a parameter limited to Cartesian co-ordinates

Partial Derivatives :

The vector differential operator ∇ , called "del" or "nabla", is **defined** in three dimensions to be:

$$\nabla=\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\,.$$

Note that these are *partial derivatives*!

This vector operator may be applied to (differentiable) scalar functions (scalar fields) and the result is a special case of a vector field, called a gradient vector field.

Here are two warming up exercises on partial differentiation.

The Del Operator or gradient :

The **gradient** of a function, $f(x,y)$, in two dimensions is defined as:

$$\operatorname{grad} f(x,y)=\nabla f(x,y)=\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}\,.$$

The gradient of a function is a vector field. It is obtained by applying the vector operator ∇ to the scalar function $f(x,y)$. Such a vector field is called a gradient (or conservative) vector field.

Example 1 The gradient of the function $f(x,y)=x+y^2$ is given by:

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}\\ &= \frac{\partial}{\partial x}(x+y^2)\boldsymbol{i}+\frac{\partial}{\partial y}(x+y^2)\boldsymbol{j}\\ &= (1+0)\boldsymbol{i}+(0+2y)\boldsymbol{j}\\ &= \boldsymbol{i}+2y\boldsymbol{j}\,.\end{aligned}$$

The definition of the **gradient** may be extended to functions defined in three dimensions, $f(x, y, z)$:

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}.$$

Directional Derivatives :

To interpret the gradient of a scalar field

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k},$$

note that its component in the $\boldsymbol{i}$ direction is the partial derivative of f with respect to x. This is the rate of change of f in the x direction since y and z are kept constant. In general, the component of ∇f in any direction is the rate of change of f in that direction.

Derivatives of a vector with respect to a parameter limited to plane polar co-ordinates

………..

Ordinary differential equations:

a) First order homogeneous differential equations

A first order homogeneous linear differential equation is one of the form

$$y' + p(t)\,y = 0$$

or equivalently $y' = -p(t)y$, where $y' = \frac{dy}{dt}$

Since first order homogeneous linear equations are separable, we can solve them in the usual way:

$$y' = -p(t)\,y$$

$$\frac{dy}{dt} = -p(t)\,y$$

$$\frac{dy}{y} = -p(t)\,dt$$

$$\int \frac{1}{y}\,dy = -\int p(t)\,dt$$

$$\ln y = -\int p(t)\,dt$$

$$\ln y = P(t) + C$$

$$y = \pm e^{P(t)+C}$$

$$y = Ae^{P(t)}$$

where $P(t) = -\int p(t)\,dt$

$A = \pm e^{C}$ is a constant.

b) Second order homogeneous differential equations with constant coefficients

Let us consider a differential equation of type

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0 \quad \text{……………………..(1)}$$

where a_2, a_1 and a_0 are some constant coefficients.

The solution is determined by supposing that there is a solution of the form $x(t) = e^{mt}$ for some value of m. When we substitute a solution of this form into (1) we get the following equation.

$$a_2 m^2 e^{mt} + a_1 m e^{mt} + a_0 e^{mt} = 0$$

$$a_2 m^2 + a_1 m + a_0 = 0 \quad \text{…………..………..(2)}$$

We now consider the possible types of solutions for (2). The first type of solution that we may get is a real root of order one, m_1. In this case we get a solution, $e^{m_1 t}$ to the differential equation. If we had two distinct such roots, $m_1 \neq m_2$, then $c_1 e^{m_1 t} + c_2 e^{m_2 t}$ would also be a solution for any constants c_1 and c_2.

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} \quad \text{……………………….(3)}$$

where c_1 and c_2 are arbitrary constants to be determined by the initial conditions of the problem.

If a solution of (2) is a single repeated root of order 2, m_1, then the solution is of the form

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} \quad \text{……………………….(4)}$$

where c_1 and c_2 are arbitrary constants to be determined by the initial conditions of the problem.

If the solution of (3) is a complex number, a + bi, then the complex conjugate, a −bi is also a solution. The solution of the differential equation is of the form

$$x(t) = e^{at}\left(C_1 \sin(bt) + C_2 \cos(bt)\right) \quad \text{…………………….(5)}$$

where c_1 and c_2 are arbitrary constants to be determined by the initial conditions of the problem.

So, we see that there are distinct types of solutions that we my have to the differential equation (1). These are given by combinations of solutions of the type (3), (4) or (5) depending on the types of roots we have for (2).

Newton’s Laws of Motion

………

Isaac Newton (a 17th century scientist) put forth a variety of laws that explain why objects move (or don't move) as they do. These three laws have become known as Newton's three laws of motion.

(a) **Newton's first law of motion (the law of inertia) :** An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an external force.

This law predicts the behaviour of stationary objects as well as moving objects. The behaviour of all objects can be described by saying that objects tend to "keep on doing what they're doing" unless acted upon by an external force. If at rest, they will continue in this same state of rest. If in motion with an eastward velocity, they will continue in this same state of motion.
Therefore, the property of a body by virtue of which it tends to remain in the same state of motion until and unless an external force is applied on it, is called the inertia of a body.
Tendency of body at rest to remain at rest, is called inertia of rest. Similarly, tendency of a body in motion to remain in motion is called inertia of motion.

(b) **Second Law of motion :**

The second law states that the acceleration of an object is dependent upon two variables - the net force acting upon the object and the mass of the object. The acceleration of an object depends directly upon the net force acting upon the object, and inversely upon the mass of the object. As the force acting upon an object is increased, the acceleration of the object is increased. As the mass of an object is increased, the acceleration of the object is decreased.

$$a \propto F$$

$$a \propto \frac{1}{m}$$

$$a \propto \frac{F}{m}$$

$$F = ma$$

where a = acceleration produced by the force F applied,   m = mass of the body

In Vector form

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = \frac{d(m\vec{v})}{dt}$$

$\vec{F} = \frac{d\vec{p}}{dt}$; where $\vec{p} = m\vec{v}$ is called the linear momentum of the body.

Therefore, the second law of Newton states that the force acting on a body to change its state of motion, is equal to the rate of change of linear momentum of the body.

(c) **Third Law of motion :**

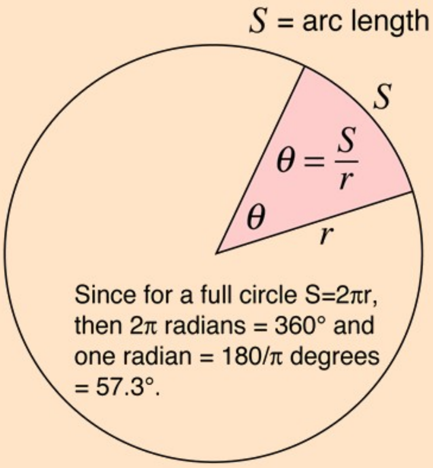
Newton's third law states that for every action, there is an equal and opposite reaction.

The statement means that in every interaction, there is a pair of forces acting on the two interacting objects. The size of the forces on the first object equals the size of the force on the second object. The direction of the force on the first object is opposite to the direction of the force on the second object. Forces always come in pairs - equal and opposite action-reaction force pairs.

Basic Rotational Quantities

The angular displacement is defined by:

$$\theta = \frac{S}{r}$$



For a circular path it follows that the angular velocity is

$$\omega = \frac{v}{r}$$

and the angular acceleration is

$$\alpha = \frac{a_t}{r}$$

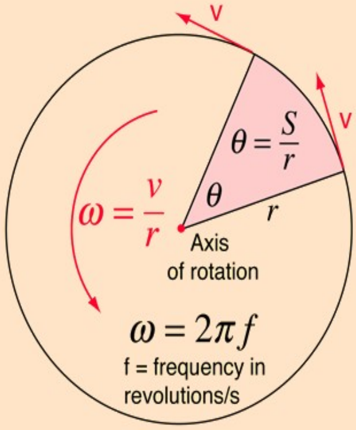
where the acceleration here is the tangential acceleration.

In addition to any tangential acceleration, there is always the centripetal acceleration:

$$a_c = \frac{v^2}{r}$$

The standard angle of a directed quantity is taken to be counterclockwise from the positive x axis.

Angular Velocity



For an object rotating about an axis, every point on the object has the same angular velocity. The tangential velocity of any point is proportional to its distance from the axis of rotation. Angular velocity has the units rad/s.

$$v = \omega r \quad or \quad \omega = \frac{v}{r}$$

Angular velocity is the rate of change of angular displacement and can be described by the relationship

$$\omega_{average} = \frac{\Delta\theta}{\Delta t}$$

and if v is constant, the angle can be calculated from

$$\theta = \theta_0 + \omega t$$

Angular velocity can be considered to be a vector quantity, with direction along the axis of rotation in the right-hand rule sense.

Vector angular velocity

Angular Momentum

Angular Momentum of a Particle

The angular momentum of a particle of mass m with respect to a chosen origin is given by

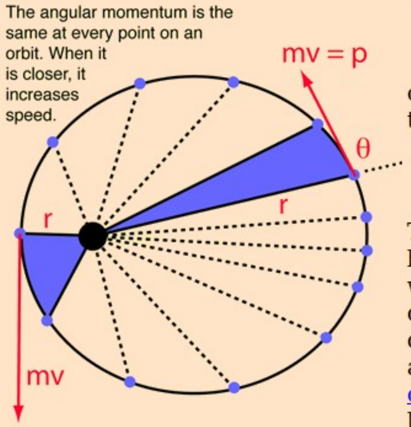
$$L = mvr \sin\theta$$

or more formally by the vector product

$$L = r \times p$$

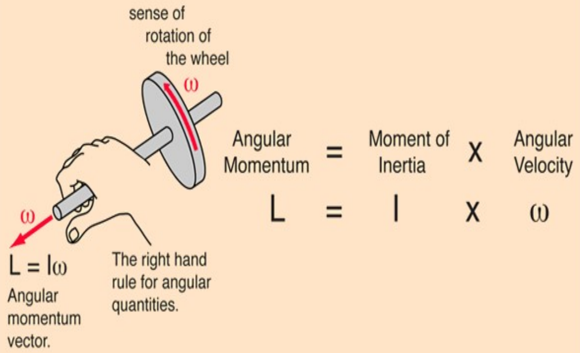
The direction is given by the right hand rule which would give L the direction out of the diagram. For an orbit, angular momentum is conserved, and this leads to one of Kepler's laws. For a circular orbit, L becomes

$$L = mvr$$



Angular Momentum

The angular momentum of a rigid object is defined as the product of the moment of inertia and the angular velocity. It is analogous to linear momentum and is subject to the fundamental constraints of the conservation of angular momentum principle if there is no external torque on the object. Angular momentum is a vector quantity. It is derivable from the expression for the angular momentum of a particle



Angular and Linear Momentum

Angular momentum and linear momentum are examples of the parallels between linear and rotational motion. They have the same form and are subject to the fundamental constraints of the conservation laws, the conservation of momentum and the conservation of angular momentum .