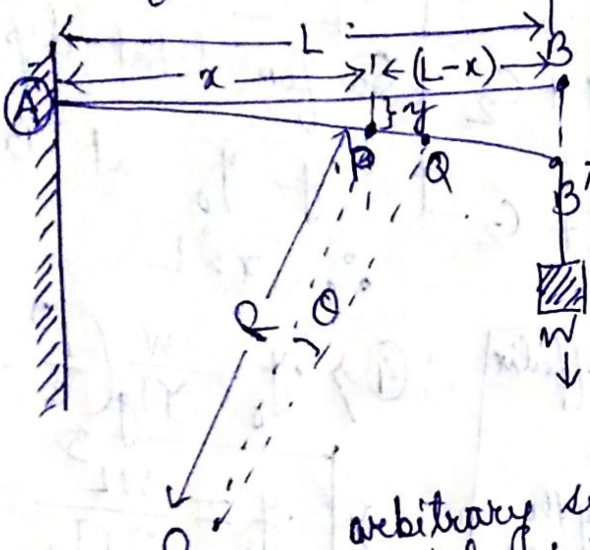
Expression for depression produced in a cantilever :

A cantilever is a beam fixed horizontally to support heavy loads.

Case (a). If the weight of the cantilever is negligible



Let us consider AB be the neutral axis of a cantilever of length L. It is fixed at A and loaded at B by a load W. As a result, the point B shifts B' through a vertical distance y_0 (total depression). Now, let consider any arbitrary section P of the cantilever at a distance x from the fixed point A.

$\therefore$ Moment of the force at P $= W(L-x)$

But bending moment of a beam (cantilever) $= \frac{YI_g}{R}$

$\therefore$ $W(L-x) = \frac{YI_g}{R} \Rightarrow \frac{1}{R} = \frac{W}{YI_g}(L-x) \longrightarrow ①$

From geometry we know, $\frac{1}{R} = \frac{\left(d^2y/dx^2\right)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$ ② ; But here slope = θ = very small

as bending is very small

$\therefore \left(\frac{dy}{dx}\right) = 0$

$\therefore$ ② $\Rightarrow \frac{1}{R} = \left(\frac{d^2y/dx^2}{1}\right) = \frac{d^2y}{dx^2}$.

Using this in ①, we have

$$\frac{d^2y}{dx^2} = \frac{W}{YI_g}(L-x)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{W}{YI_g}(L-x)$$

$$\Rightarrow d\left(\frac{dy}{dx}\right) = \frac{W}{YI_g}(L-x)dx$$

Integrating both sides

$$\int d\left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\int(L-x)dx + C_1$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right) + C_1 \rightarrow ③$$

At $x=0, \frac{dy}{dx}=0$, $c_1=0$

$\therefore$ (3) $\Rightarrow \left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right)$

$\Rightarrow dy = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right)dx$

Again integrating both sides

$$\int dy = \int \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right)dx + C_2$$

$$\Rightarrow y = \frac{W}{YI_g}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + C_2$$

at $x=0$, $y=0$

$\therefore C_2 = 0$

$$\therefore \boxed{y = \frac{W}{YI_g}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right)} \quad (4)$$

So for total depression,

$y = y_0$ at B.

$\therefore x = L$

$\therefore$ (4) $\Rightarrow y_0 = \frac{W}{YI_g}\left(\frac{L^3}{2} - \frac{L^3}{6}\right)$

$$\Rightarrow \boxed{y_0 = \frac{WL^3}{3YI_g}} \rightarrow (5)$$

Case b: When weight of the cantilever is effective

[Description same as case a]

Let w be the weight per unit length of the cantilever, and this weight also produces bending in it, in addition to W.

$\therefore$ Moment of force at P is $= W(L-x) + w(L-x)\frac{(L-x)}{2}$

$= W(L-x) + w\frac{(L-x)^2}{2}$

$$\therefore W(L-x) + w\frac{(L-x)^2}{2} = \frac{YI_g}{R} \longrightarrow (6)$$

$$\Rightarrow \frac{1}{R} = \frac{W}{YI_g}(L-x) + \frac{w}{YI_g}\frac{(L-x)^2}{2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{W}{YI_g}(L-x) + \frac{w}{YI_g}\frac{(L-x)^2}{2}$$

Integrate both sides

$$\int d\left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\int(L-x)dx + \frac{w}{2YI_g}\int(L-x)^2 dx + C_1$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right) + \frac{w}{2YI_g}\left[L^2x + \frac{x^3}{3} - 2L\frac{x^2}{2}\right] + C_1$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right) + \frac{\omega}{2YI_g}\left[L^2x + \frac{x^3}{3} - Lx^2\right] + C_1$$

at $x=0$, $\frac{dy}{dx}=0$, $C_1=0$

$$\therefore \left(\frac{dy}{dx}\right) = \frac{W}{YI_g}\left(Lx - \frac{x^2}{2}\right) + \frac{\omega}{2YI_g}\left(L^2x + \frac{x^3}{3} - Lx^2\right).$$

Again integrating both sides

$$\int dy = \frac{W}{YI_g}\int\left(Lx - \frac{x^2}{2}\right)dx + \frac{\omega}{2YI_g}\int\left(L^2x + \frac{x^3}{3} - Lx^2\right)dx + C_2$$

$$\Rightarrow y = \frac{W}{YI_g}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + \frac{\omega}{2YI_g}\left(L^2\frac{x^2}{2} + \frac{x^4}{12} - L\frac{x^3}{3}\right) + C_2$$

At $x=0$, $y=0$, $C_2=0$

$$\therefore y = \frac{W}{YI_g}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + \frac{\omega}{2YI_g}\left(L^2\frac{x^2}{2} + \frac{x^4}{12} - L\frac{x^3}{3}\right) \longrightarrow (7)$$

For total depression at B, $y = y_0$, $x = L$

$$\therefore (6) \Rightarrow y_0 = \frac{W}{YI_g}\frac{L^3}{3} + \frac{\omega}{2YI_g}\left(\frac{L^4}{2} + \frac{L^4}{12} - \frac{L^4}{3}\right)$$

$$= \frac{W}{YI_g}\frac{L^3}{3} + \frac{\omega}{2YI_g}\left(\frac{L^4}{4}\right)$$

$$= \frac{W}{YI_g}\frac{L^3}{3} + \frac{\omega}{YI_g}\frac{L^4}{8}$$

$$= \frac{WL^3}{3YI_g} + \frac{(\omega L)L^3}{8YI_g}$$

$$= \frac{WL^3}{3YI_g} + \frac{W_1L^3}{8YI_g}$$; $W_1 = \omega L$ is the total ~~length~~ weight of the beam

$$\boxed{y_0 = \left(W + \frac{3}{8}W_1\right)\frac{L^3}{3YI_g}} \rightarrow (8)$$

Case (c) When no external weight is applied and only the weight of the beam itself is producing bending.

[Description same as case (a)] but $W = 0$

$\therefore$ If ω is the weight of the cantilever per unit length, then (6) (8) $\Rightarrow$ $\omega(L-x)\frac{(L-x)}{2} = \frac{YI_g}{R} \longrightarrow$ (9)

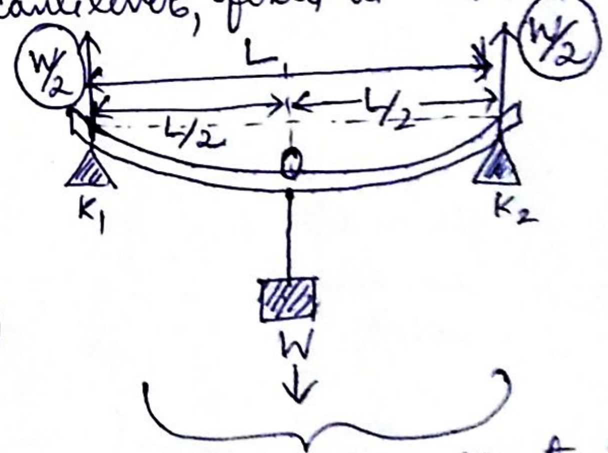
$$\Rightarrow \frac{1}{R} = \frac{\omega}{2YI_g}(L-x)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\omega}{2YI_g}(L-x)^2$$

Integrating twice (write the procedure as case (b))

$$\boxed{y_0 = \frac{W_1 L^3}{8YI_g}} \; ; \; W_1 = \omega L \longrightarrow (10)$$

Case d Depression produced in a cantilever, fixed at both ends and loaded in the middle



Therefore, for this case

$$W \to \frac{W}{2}, \quad L \to \frac{L}{2}$$

$\therefore$ Total depression (from case (a))

$$y_0 = \frac{W L^3}{3YI_g}$$

$$\Rightarrow y_0 = \frac{\left(\frac{W}{2}\right)\left(\frac{L}{2}\right)^3}{3YI_g}$$

$$\Rightarrow \boxed{y_0 = \frac{WL^3}{48YI_g}}$$

It is the combination of two beams fixed at one point and loaded at the other.

[Repeat the Case (a)]