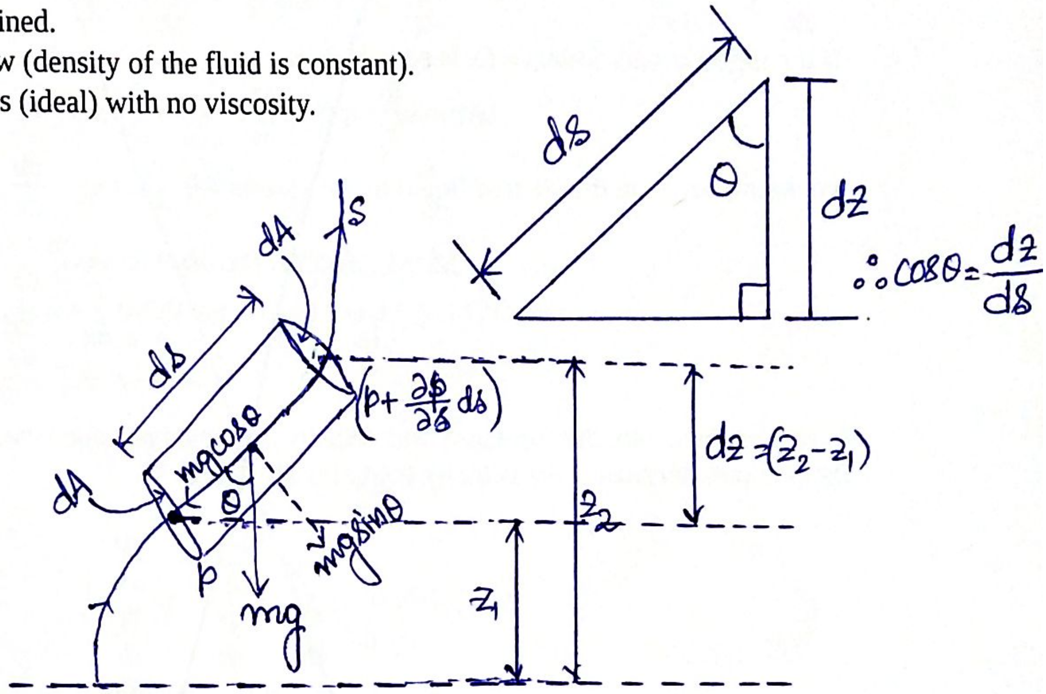
Euler's Equation of Motion for fluid flow

For deriving the Euler's Equation for liquid flow, let us consider the flow of a fluid which is

- Steady and streamlined.
- Incompressible flow (density of the fluid is constant).
- Fluid is non viscous (ideal) with no viscosity.



Let us consider a small element of the fluid flow of length *ds*. The pressure at the point A is *p* and it becomes $p + \frac{dp}{ds}ds$, at the point B.

Now,

Resultant force on the element = $p.dA - (p + \frac{dp}{ds}ds).dA - mg\cos\theta$

$$= -\frac{dp}{ds}ds.dA - (\rho.ds.dA)g\cos\theta$$

But, *Resultant Force = Mass. Acceleration*

$$-\frac{dp}{ds}ds.dA - (\rho.ds.dA)g\cos\theta = (\rho.ds.dA).a_s \quad \text{.................................. (1)}$$

But here, $v_s = f(r, t) = f(s,t)$; $a_s = \frac{dv}{dt} = \frac{\delta v}{\delta s}\frac{\delta s}{\delta t} + \frac{\delta v}{\delta t}$

For steady flow, $\frac{\delta v}{\delta t} = 0$

So, $a_s = \frac{dv}{dt} = \frac{\delta v}{\delta s}\frac{\delta s}{\delta t} = v\frac{dv}{ds}$

Using this in (1), we have

$$-\frac{dp}{ds}ds.dA - (\rho.ds.dA)g\cos\theta = (\rho.ds.dA).v.\frac{dv}{ds}$$

$$-\frac{dp}{ds} - \rho . g\cos\theta = \rho . v . \frac{dv}{ds}$$

$$\frac{dp}{ds} + \rho . g\cos\theta + \rho . v . \frac{dv}{ds} = 0$$

$$\frac{1}{\rho} . \frac{dp}{ds} + g\cos\theta + v . \frac{dv}{ds} = 0$$

But , $\cos\theta = \frac{dz}{ds}$ (from geometry)

$$\frac{1}{\rho} \frac{dp}{ds} + g \frac{dz}{ds} + v . \frac{dv}{ds} = 0$$

$$\frac{dp}{\rho} + g\, dz + v\, dv = 0 \quad \text{.................................(2)}$$

which is the **Euler's equation** for fluid motion.

Now, integrating both sides of (2), we have

$$\int_{p_1}^{p_2} \frac{dp}{\rho} + \int_{z_1}^{z_2} h\, dz + \int_{v_1}^{v_2} v\, dv = 0$$

$$\frac{1}{\rho}(p_2 - p_1) + h(z_2 - z_1) + \frac{1}{2}(v_2^2 - v_1^2) = 0$$

$$\frac{p_1}{\rho} + hz_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + hz_2 + \frac{1}{2} v_2^2$$

$$\frac{p}{\rho} + hz + \frac{1}{2} v^2 = Constant$$

Pressure Energy per unit mass + Potential Energy per unit mass + Kinetic Energy per unit mass = Constant

which is statement for Bernoull's Theorem.

Bernoulli's Theorem:

It states that the total amount of energy per unit mass of a non-viscous incompressible liquid flowing from one point to another, without any friction, remains constant throughout the displacement.

It follows that, in any streamline flow of liquid, the loss of energy in one form is equal to the gain of energy in another form.

i.e. **Potential Energy + Pressure Energy + Kinetic Energy = Constant**

This relation is known as ***Bernoulli's Theorem.***