Gauss Elimination Method :

This method is based on the idea of reducing the given system of equations $Ax = b$, to an upper triangular system of equations $Ux = Z$, using elementary row operations. These two systems are equivalent, so their solutions are identical.

Let us consider the system of equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \longrightarrow (1)$$

ie $AX = b$

Now, the augmented matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \longrightarrow (2)$$

So, under this method, we need to do the following transformation.

$$[A|b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U|Z]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

First stage elimination :

Let us assume that $a_{11} \neq 0$. It is called the first pivot.

Now, we will perform the following transformations.

(i) Multiply the first row of ② by $\frac{a_{21}}{a_{11}}$ and substract the value from the corresponding values on 2nd column row.

(ii) Multiply the first row of ② by $\frac{a_{31}}{a_{11}}$ and substract the value from the corresponding values on 3rd row.

$\left[R_2 - \left(\frac{a_{21}}{a_{11}}\right)R_1\right] \rightarrow$

$$\therefore \left.\begin{aligned} (a_{21})^{(1)} &= a_{21} - a_{11}\left(\frac{a_{21}}{a_{11}}\right) = 0 \\ (a_{22})^{(1)} &= a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right) \\ (a_{23})^{(1)} &= a_{23} - a_{13}\left(\frac{a_{21}}{a_{11}}\right) \end{aligned}\right\} \quad (b_2)^{(1)} = b_2 - \left(\frac{a_{21}}{a_{11}}\right)b_1$$

First elimination (i)

$\left[R_3 - \left(\frac{a_{31}}{a_{11}}\right)R_1\right] \rightarrow$

$$\left.\begin{aligned} (a_{31})^{(1)} &= a_{31} - a_{11}\left(\frac{a_{31}}{a_{11}}\right) = 0 \\ (a_{32})^{(1)} &= a_{32} - a_{12}\left(\frac{a_{31}}{a_{11}}\right) \\ (a_{33})^{(1)} &= a_{33} - a_{13}\left(\frac{a_{31}}{a_{11}}\right) \\ b_3^{(1)} &= b_3 - \left(\frac{a_{31}}{a_{11}}\right) b_1 \end{aligned}\right\}$$

First Elimination (ii)

$\therefore$ New augmented matrix is

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right] \longrightarrow ③$$

Second Stage Elimination :

Now, assume that $a_{22}^{(1)} \neq 0$ and this element is called second pivot.

(i) Multiply the 2nd row in ③ by $\frac{a_{32}^{(1)}}{a_{22}^{(1)}}$ and substract the value from ~~the~~ the corresponding values on the third row. ie

$$R_3 \to \left[R_3 - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} R_2\right]$$

∴ New augmented matrix is

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right] \longrightarrow ④$$

$$\therefore \left.\begin{aligned} a_{33}^{(2)} &= a_{33}^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} a_{23}^{(1)} \\ b_3^{(2)} &= b_3^{(1)} - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} b_2^{(1)} \end{aligned}\right\} \text{2nd stage elimination}$$

The element $a_{33}^{(2)} \neq 0$, is called the third pivot.

So, the system ④ is in upper triangular form.
Now, doing back substitution, we have

$$\therefore \left.\begin{aligned} x_3 &= \frac{b_3^{(2)}}{a_{33}^{(2)}} & ⓐ \\ x_2 &= \left(b_2^{(1)} - a_{23}^{(1)} x_3\right)/a_{22}^{(1)} & ⓑ \\ x_1 &= \left(b_1 - a_{12}x_2 - a_{13}x_3\right)/a_{11} & ⓒ \end{aligned}\right\} ⑤$$

∴ Doing back substitution for the solution of x, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} \\ a_{22}^{(1)} x_2 + a_{23}^{(1)} x_2 \\ a_{33}^{(2)} x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{pmatrix}.$$

$$\therefore \quad x_1 a_{11} + x_2 a_{12} + x_3 a_{13} = b_1$$

$$\Rightarrow \boxed{x_1 = (b_1 - x_2 a_{12} - x_3 a_{13}) / a_{11}}$$

$$a_{22}^{(1)} x_2 + a_{23}^{(1)} x_3 = b_2^{(1)}$$

$$\Rightarrow \boxed{x_2 = \left(b_2^{(1)} - a_{23}^{(1)} x_3\right) / a_{22}^{(1)}}$$

$$a_{33}^{(2)} x_3 = b_3^{(2)}$$

$$\Rightarrow \boxed{x_3 = \frac{b_3^{(2)}}{a_{33}^{(2)}}}$$

Pivoting Procedures :

(i) Partial pivoting : in the first stage of elimination, the first column of the augmented matrix is searched for the largest element in magnitude and brought as the first pivot by interchanging the first row of the augmented matrix of first equation by the row, having the largest element in magnitude. In the second stage of elimination, the second column is searched for the largest element in magnitude among the $(n-1)$ elements leaving the first element and this element is brought as the second pivot by interchanging the second row of the new augmented matrix with the row having largest element in magnitude. This process is continued till the upper triangular matrix is obtained. This process is called partial pivoting.

(ii) Complete pivoting : In this process, we search the entire matrix A in the augmented matrix for the largest element in magnitude and bring it as the first pivot. This requires interchange of ~~both~~ rows and ~~column~~ interchange of the positions of the variables. Here we need to keep track of the changes in the positions of the variable. Hence this process is computationally expensive and not used in any software.