Derivative using Newton's Backward Difference formula :

Let us consider the data $(x_i, f(x_i))$ given at equispaced points $x_i = x_0 + ih$, where h is the step length.

The Newton's Backward difference formula is given by

$$f(x) = f(x_n) + (x-x_n)\frac{1}{\lfloor 1\, h}\nabla f(x_n) + (x-x_n)(x-x_{n-1})\frac{1}{\lfloor 2\, h^2}\nabla^2 f(x_n)$$

$$+ (x-x_n)(x-x_{n-1}) \cdots (x-x_1)\frac{1}{\lfloor n\, h^n}\nabla^n f(x_n) \longrightarrow (1)$$

Let us assume that, x is a point near x_n.

$$\therefore \quad x - x_n = sh$$

$$\therefore (1) \Rightarrow f(x) = f(x_n + sh)$$

$$= f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{\lfloor 2}\nabla^2 f(x_n)$$

$$+ \frac{s(s+1)(s+2)}{\lfloor 3}\nabla^3 f(x_n) + \cdots + \frac{s(s+1)(s+2)\cdots(s+n-1)}{\lfloor n}\nabla^n f(x_n) \longrightarrow (2)$$

Here $\quad s = \left(\frac{x-x_n}{h}\right) < 0.$

So, magnitude of the successive terms on the right ~~and~~ hand side become smaller and smaller.

Now, differentiating (2)

$$\frac{df}{dx} = \frac{df}{ds}\frac{ds}{dx}$$

$$\Rightarrow f'(x_n) = \frac{1}{h}\frac{df}{ds}$$

$$= \frac{1}{h}\Big[\nabla f_n + \frac{1}{2}(2s+1)\nabla^2 f_n + \frac{1}{6}(3s^2+6s+2)\nabla^3 f_n$$
$$+ \frac{1}{24}(4s^3+18s^2+22s+6)\nabla^4 f_n$$
$$+ \frac{1}{120}(5s^4+40s^3+105s^2+100s+24)\nabla^5 f_n\Big] \longrightarrow (3)$$

At $x = x_n$, $s = 0$.

$$\therefore f'(x_n) = \frac{1}{h}\Big[\nabla f_n + \frac{1}{2}\nabla^2 f_n + \frac{1}{3}\nabla^3 f_n + \frac{1}{4}\nabla^4 f_n + \frac{1}{5}\nabla^5 f_n\Big] \longrightarrow (4)$$

At $x = x_{n-1}$, $x_{n-1} = (x_n - h)$

$$= x_n + sh$$

$$\therefore s = (-1)$$

$$\therefore f'(x_{n-1}) = \frac{1}{h}\Big[\nabla f_n - \frac{1}{2}\nabla^2 f_n - \frac{1}{6}\nabla^3 f_n - \frac{1}{12}\nabla^4 f_n - \frac{1}{2}\nabla^5 f_n + \cdots\Big] \longrightarrow (5)$$

Again
$$\frac{d^2 f}{dx^2} = \frac{1}{h^2}\frac{d}{ds}\left(\frac{df}{ds}\right)$$

$$= \frac{1}{h^2}\Big[\nabla^2 f_n + \frac{1}{6}(6s+6)\nabla^3 f_n +$$
$$\frac{1}{24}(12s^2+36s+22)\nabla^4 f_n + \frac{1}{120}(20s^3+120s^2$$
$$+210s+100)\nabla^5 f_n + \cdots\Big] \rightarrow (6)$$

At $x = x_n$; $s = 0$

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 f_n + \nabla^3 f_n + \frac{11}{12}\nabla^4 f_n + \frac{5}{6}\nabla^5 f_n + \frac{137}{180}\nabla^6 f_n + \cdots\right] \longrightarrow (7)$$

At $x = x_{n-1}$, $s = -1$.

$$\therefore f''(x_{n-1}) = \frac{1}{h^2}\left[\nabla^2 f_n - \frac{1}{12}\nabla^4 f_n - \frac{1}{12}\nabla^5 f_n + \cdots\right] \longrightarrow (8)$$